Classical Circuit Theory

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To all my students, worldwide

Preface

Classical circuit theory is a mathematical theory of linear, passive circuits, namely, circuits composed of resistors, capacitors and inductors. Like many a thing classical, it is old and enduring, structured and precise, simple and elegant. It is simple in that everything in it can be deduced from first principles based on a few physical laws. It is enduring in that the things we can say about linear, passive circuits are universally true, unchanging. No matter how complex a circuit may be, as long as it consists of these three kinds of elements, its behavior must be as prescribed by the theory. The theory tells us what circuits can and cannot do.

As expected of any good theory, classical circuit theory is also useful. Its ultimate application is circuit design. The theory leads us to a design methodology that is systematic and precise. It is based on just two fundamental theorems: that the impedance function of a linear, passive circuit is a positive real function, and that the transfer function is a bounded real function, of a complex variable.

In this book, we begin with basic principles of circuits, derive their analytic properties in both the time and frequency domains, and state and prove the two important theorems. We then develop an algorithmic method to design common and uncommon types of circuits, such as prototype filters, lumped delay lines, constant phase difference circuits, and delay equalizers. Along the way, we learn about the relation between gain and phase, linear and minimum phase functions, group delay, sensitivity functions, scattering matrix, synthesis of transfer functions, approximation of filter functions, all-pass circuits, and circuit design by optimization.

The book is written as a text suitable for use by seniors or first year graduate students in a second course in circuit theory. It can be covered in one semester or one quarter at a brisk pace. Chapter Two, which is on fundamentals, and Chapter Three, which is on circuits in the time domain, can be omitted, if the principal aim of the course is circuit design.

In this age of digital signal processing and analog electronic filters, one may ask why we want to study passive filter design. The reason is that the operating frequency of a digital filter is limited by how fast we can sample an input signal and by how fast we can process the signal samples digitally. While digital circuitry has made steady improvement in its operating speed, it is still too slow for some circuits used in many wireless communication systems.

As to analog electronic filters, scores of configurations have been proposed over the years. The conventional design is to realize a transfer function as a cascade of second order sections, each of which is implemented in a circuit of resistors, capacitors, integrators and summers. More recently, it is found that a better methodology is to first realize a passive filter as a lossless (inductor-capacitor) ladder terminated in resistors, and then to simulate the passive filter with integrators and summers. The new configuration has superior sensitivity properties. What all this means is that we need to know how passive filters are designed in the first place.

Similarly, a microwave filter is usually first designed as a passive filter with inductors and capacitors. Then each inductor is replaced by a length of transmission line of certain characteristic impedance and each capacitor by another line of different length and characteristic impedance. More commonly, the inductor-capacitor resonant sub-circuits in a passive filter are replaced by microwave resonators connected by short lengths of transmission lines. The basis of design is again classical circuit theory.

Modern textbooks on analog electronic filters and microwave filters abound, but not those on classical circuit theory. It seems there is a need for a modern text on the mathematical foundations of passive circuit analysis and design, which are what classical circuit theory is all about. Classical circuit theory is old, but it has survived the test of time and it is still relevant today because it is basic.

In writing this book, I have benefited from feedback from students at Columbia University, where the draft of the book had been classroom-tested three times. Their comments had been helpful and are hereby gratefully acknowledged.

I also want to thank the Department of Electrical Engineering, Columbia University, where I taught circuit theory for thirty-five years, for its continual support. In addition, I am grateful to the following institutions for having invited me to teach various aspects of circuit theory to a collective group of outstanding students. The institutions are: Chiao Tong University, Taiwan (1961); The Technical University of Denmark (1973); Indian Institute of Technology, Kanpur (1978); South China University of Technology (1979); Eindhoven University of Technology (1979-80); Shanghai Jiao Tong University (1982, 2000); East China University of Technology (1985, 1990); Beijing Post and Telecommunications University (1990); The Chinese University of Hong Kong (1991-98); Fudan University, Shanghai (2001, 2003); and Columbia Video Network (2005).

A book is the work of its author. But to craft it into an endearing product requires the assistance of an experienced editorial and production staff. The staff at Springer played this role, to whom I am most grateful.

Pomona, New York *Omar Wing*

June 2008

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Chapter 1
Introduction

We begin with a brief history of circuit theory. It is always good to learn from the past, to see how the pioneers discovered the natural laws, principles, analysis and synthesis techniques that form the body of circuit theory. It is also good to know what motivated the development of circuit theory and what applications are driving circuit theory today. In this introductory chapter, we touch on these topics and give an overview of the scope of this book.

1.1 A brief history

It is generally agreed that circuit theory began with the formulation, in 1845, of Kirchhoff's current and voltage laws, which set forth the equilibrium conditions of the currents and voltages that may exist in a circuit. The pioneering work, in the early 1800s, of Volta, Ampère, Ohm, Faraday, Henry, Siemens, and later Maxwell led to laws that define the current-voltage relations of circuit elements, which at that time were the resistor, inductor, coupled inductors and capacitor. Kirchhoff's laws, together with the definitions of circuit elements, constitute the foundation of circuit theory. Everything about circuits: analysis methods, analytic properties, theoretical limitations, design techniques, can be derived from first principles based on these laws and definitions.

It was Maxwell, in 1881, who put circuit analysis on a formal, mathematical basis. He introduced node equations and mesh equations to describe circuits by a set of maximally independent linear equations. Though mesh equations are applicable only to planar circuits, node equations, with modification that came later, can be used to describe any circuit and are the ones adopted in all circuit simulation programs today.

As circuits grew in size and complexity (at least by 19th century standards), the idea of "equivalent" circuits as a means to simplify circuit analysis became attractive. Thévenin showed in 1883 that a linear circuit across a pair of terminals, or a source of electricity, whose internal composition may not be known, can be represented by an equivalent circuit consisting of a single voltage source in series with a resistor or an impedance. In 1926, Norton extended the idea to a representation consisting of a current source in parallel with an impedance.

When telegraphy came, there was a need to compute the transient response of circuits when the excitation was a pulse. Heaviside, in 1880-87, introduced operational calculus for this purpose, which also led to the representation of voltages and currents as complex variables in the transformed domain. To extend the useful range of telegraphy and telephony over long distances, Heaviside and later Pupin and Campbell proposed the use of "loading coils" periodically inserted along the transmission lines (see Problems 4.22 and 8.21). Their theoretical studies paid off handsomely, at least for Pupin.[1] Campbell went on to develop a filter design methodology known as "wave filter" or "image parameter" theory, which was a practical way to arrive at a good design quickly. Although the method is one by trial and error, it had been used to design just about all the filters and equalizers for telephone networks during the period when carrier telephony was expanding in the U.S. and elsewhere in the first half of the 20th century.

As alternating current became the standard mode of generating and distributing electricity at the turn of the 20th century, Steinmetz came up with the idea of using complex numbers to represent voltages and currents in the sinusoidal steady state. The concepts of impedance, transfer function, magnitude and phase came into being, and circuits can be analyzed entirely in the frequency domain using complex algebra.

Treating circuit functions as functions of a complex variable, Foster [26], Cauer [16], and Bode [8], in the 1920s and 1930s, derived many of the fundamental properties of circuits important to the design of filters, delay equalizers and frequency selective circuits used in communications systems. Such properties as the relations between gain and phase, gain-bandwidth trade-off, stability of feedback circuits, were discovered principally by Bode around this time.

It was Brune [12], in 1931, who found the necessary and sufficient conditions for a rational function to be an impedance function of a circuit consisting of inductors (including ideal transformers), capacitors and resistors. He showed that a rational function can be realized as the input impedance of such a circuit if and only if it is a positive real function. (It is real when its argument is real; its real part is positive when the real part of its argument is positive.) The use of ideal transformers is not desirable and in 1949, Bott and Duffin [9] found a realization of a positive real function as the impedance of a circuit composed of only resistors, capacitors and inductors, without transformers. Since then we have not been able to come up with another transformerless realization.

Inspired by Brune's work, Darlington [19] in 1939 derived the necessary and sufficient conditions for a rational function to be realizable as the transfer function of a lossless two-port terminated in a one-ohm resistor. At about the same time, Cauer [16] and Belevitch [4] in Europe, Fujisawa [27] in Japan, and possibly others, developed similar results. They ushered in what was called the "insertion loss"

[1] Pupin received what was then a substantial amount of money from AT&T for the right to use his patents.

design of filters. For the first time, filter design was no longer a trial and error affair. Given the specifications of a prototype filter characteristics, one obtains a realizable transfer function, which is then realized systematically as a lossless two-port terminated in a resistor at the input and output ends. However, this design methodology was not widely adopted by practitioners until the digital computer was available to provide the necessary computation power and the required precision of the element values. Design tables were then compiled to give the circuit configurations and element values of practical, prototype filters with maximally flat pass-band, equal-ripple pass-band, and equal-ripple pass and stop band characteristics. Today, filter design algorithms have been incorporated in commercial software and after the desired frequency characteristics have been entered, the design is but one click away. The filter prototypes designed by the insertion loss method have one additional desirable property, and it is that the sensitivities of the loss characteristics with respect to element variations are zero at the frequencies where the loss is zero, usually in the pass-band.

In addition to filter design, the insertion loss theory was found to be useful in the design of broadband matching circuits. Such a circuit is inserted between a signal source and a fixed load impedance to maximize power transfer to the load over the pass-band. The problem had its origin in an attempt to extend the useful bandwidth of telegraph and telephone cables but was only formalized and solved recently by Fano [23, 24], Youla [70], Carlin [14], Chen [17], and others for the general case.

Darlington's synthesis in general requires the use of ideal transformers, which are not practical. In actual applications, the prototype filters are restricted to be two-ports that can be realized as ladder circuits composed of inductors and capacitors. But inductors are also not practical, being bulky, lossy and expensive to manufacture with precision. When vacuum tubes, then transistors, and more recently integrated circuits came along, attempts were made to replace the inductor with a combination of operational amplifiers and capacitors. Over the years, hundreds, if not thousands, designs have been proposed to realize a general transfer function with resistors, capacitors and operational amplifiers, and the age of *RC* active filter was born, in about 1960. As a result, transfer function synthesis amounts to factoring a suitable rational function into first and second order rational functions, and each is realized by a first or second order active filter section. The component sections are then cascaded to realize the overall transfer function. Most of the filters in use today are active filters.

Active filters essentially have set aside the need to study synthesis of circuits composed of passive elements. Although in recent years, a passive prototype is first designed and it is then simulated by a circuit of integrators and summers, in much the same way as its state equations are implemented on an analog computer.

Active filters are basically relatively low-frequency, small signal circuits. The highest signal frequency must be below the normal operating frequency of the active devices, and the signal amplitude throughout a filter must be small compared to the DC biases of the active elements, lest nonlinear distortion sets in. Within these limitations, active filters are analyzed in the same way as linear circuits, and many of the analytic properties such as gain-phase relations, gain-bandwidth tradeoffs, stability conditions, are valid for both.

All electronic circuits are nonlinear. Indeed it is the nonlinear properties of the transistors that are utilized to construct communication circuits such as modulators, mixers, oscillators, and detectors, and digital circuits such as AND gates and flip-flops. Unfortunately, it is not possible to obtain circuit responses in closed form and as a result, it is difficult to deduce general principles. Each circuit type must be treated individually. Nonlinear circuit theory becomes a study of circuit properties and analysis techniques of a collection of special cases.

Looking back at the most recent past, we see substantial efforts were made to develop computation algorithms to simulate and to design very-large-scale integrated (*VLSI*) circuits, both small-signal and large-signal, linear and nonlinear. Indeed the popularization of the personal computer, the Internet, the cellular phone and personal entertainment devices, owes much to the work of circuit theorists who developed efficient and reliable computation tools to help engineers design complex circuits that "work the first time." Their theoretical investigations included modeling of lossy interconnects of integrated circuits, and numerical solution of nonlinear ordinary differential equations whose linearized equivalent has widely dispersed eigenvalues in the complex plane.

The next frontier, as far as circuit theory is concerned, seems to be the design of circuits that operate in the GHz or even TeraHz range. It is not clear if it is still possible to identify discrete circuit elements, but we know the electronic devices are there and the needs for such circuits are there.

1.2 What drives circuit theory?

In one word: communications. It was the need to extend the useful range of long-distance telegraphy and telephony that spurred the invention of the loading coil and initiated the theoretical study of broadband matching techniques, filter design and modeling of lossy transmission lines. Radio motivated the study of nonlinear oscillation. Trans-Atlantic telephony over submarine cables and transcontinental telephone and TV services over coaxial cables needed repeaters that were stable, long-lasting, and immune to noise. This led to the study of feedback amplifiers and the derivation of theoretical limitations on gain-bandwidth trade-offs. Carrier telephony required banks of closely spaced bandpass filters with sharp cutoff and linear phase characteristics and this motivated the investigation of the relations between gain and phase, and the study of algorithmic design of filters and delay equalizers. The need in radar and sonar systems to provide delay to a signal spurred the design of lumped circuits to approximate a lossless transmission line. This problem amounts to approximating an irrational function with essential singularities with a rational function of finite degree. More recently, the same problem exists in modeling interconnects of integrated circuits and it has led to the development of theory of *RC* distributed parameter circuits[52, 53].

The advent of digital communications has created a need of different kind. Digital signal processing requires digital filters which are routinely designed with digital multipliers, adders and delay elements. As long as the signal frequency is not too high, digital circuits exist to provide all the necessary processing functions, and digital filter design has become a new field of study quite separate from traditional circuit theory. However, nature's signals seem to be all analog, and before they are digitized, they are usually processed with analog circuits for spectral shaping to minimize inter-symbol interference.

Personal communications with the aim of providing voice and data services to anyone, anytime, any place has created a need to design narrow-band circuits with low-Q elements embedded in highly nonlinear circuits which make up the transceiver. To increase the apparent Q, negative resistance circuits have been proposed and a good theory is needed to understand and help design such circuits.

The foregoing review demonstrates that communications has always been the driving force behind the advancement of circuit theory. The end is not near. As long as there are needs for faster ways to transmit information from one place to another, there will always be needs for new communications systems and circuits.

1.3 Scope of this book

Broadly speaking, there are three classes of circuits: linear circuits composed of passive elements, small-signal active circuits composed of resistors, capacitors and transistors, and nonlinear large-signal circuits including digital circuits and communication circuits. Over the years, each class has developed its own method of analysis and its own design techniques. From a pedagogical point of view, it is best to study these three classes separately. In this book, we take up the first class, though from time to time, we make reference to the other classes to highlight the differences. It must be said that since the 1930s, scores of theorems, principles, and special techniques of analysis have been discovered, but only a few have been found to be useful and have survived the test of time. We shall be selective in what to include in this book. The book is meant to be a text and not an encyclopedia of circuit theory.

The following are the main topics of the book:

1. Fundamentals: Starting with Kirchhoff's laws, we prove Tellegen's theorem [63], which turns out to be extremely useful in deriving properties of impedance and transfer functions. We introduce the modified node equations to describe circuits and show how to solve them numerically in the time domain.
2. Circuit dynamics: Here we introduce the state space description of circuits and derive the general solution of circuits in the time domain for any excitation in terms of the convolution integral.
3. Frequency domain analysis: Some of the fundamental properties of circuits in the frequency domain are derived here. Among them are the gain and phase relations, relations between the real and imaginary parts, phase and group delay. The

significance of minimum phase and linear phase is explained. We also derive formulas to compute sensitivity and group delay without differentiation.
4. Impedance functions: We show that an impedance function of an *RLC* circuit is a positive real function and derive its necessary and sufficient conditions.
5. Synthesis of LC, *RC* and *RL* impedance functions: The four classes of prototype realizations are presented as are the necessary and sufficient conditions for an impedance function to be any of the three two-element kind impedances.
6. Synthesis of *RLC* impedance functions: We show that every positive real function can be realized as the input impedance of an *RLC* circuit with or without transformers.
7. Two-port characterizations: Here we take up scattering matrix as a terminal description of a two-port. Other useful characterizations are also introduced.
8. Transfer function synthesis: We restrict ourselves to synthesis of a rational function that can be realized as the transfer function of a lossless two-port terminated at both ends in resistances. We further limit the two-port to be an *LC* ladder, for practical reasons.
9. Filter functions: The transfer functions of the prototype low-pass filters with Butterworth, Chebyshev, and Cauer characteristics are derived, as are their respective design formulas. That these filters have zero sensitivity at the frequencies of zero loss in the passband is demonstrated. Frequency transformations to convert a low-pass filter to a high-pass, band-pass or band-elimination filter are introduced.
10. Circuit design by optimization: Circuit design is formulated as a minimization problem and basic optimization techniques are introduced. Examples are given to illustrate how the techniques can be used to design non-standard low-pass filters and broadband matching networks with complex load impedances.
11. All-pass circuits: Constant-resistance and non-constant-resistance all-pass circuits are introduced. We show how they can be used in the design of lumped delay lines, 90° phase difference circuits in single-sideband systems, and delay equalizers to correct the phase of a filter so that the overall delay is constant in the passband.

1.4 Mathematical programming tools

To ease the computational chore, which could be substantial in filter design, readers are encouraged to use mathematical programming tools such as MATLAB®[2] in doing their problems and exercises. Useful MATLAB® functions will be introduced as they are needed and are listed in Appendix A .

[2] MATLAB® is a trademark of The MathWorks, Inc.

1.5 Notable people in classical circuit theory

To conclude this introductory chapter, we give a partial list of persons who have made significant and lasting contributions to classical circuit theory. It is not possible to include everyone. Any omission is not intentional.

1. Alessandro Volta (1745-1827). Italian physicist. Inventor of the electric battery. The unit of potential *volt* is named after him. He also studied the relation between potential and charge.
2. Andrè-Maria Ampère (1775-1836). French physicist. He formulated the mathematical relations between electric and magnetic fields. The unit of current *ampere* is named after him.
3. Georg Simon Ohm (1789-1854). German physicist. He showed the current in a conductor was proportional to the cross-section area and inversely proportioned to its length. The unit of resistance *ohm* is named after him.
4. Michael Faraday (1791-1867). English chemist and physicist. He studied electromagnetic induction and rotation. The unit of capacitance *farad* is named after him.
5. Joseph Henry (1797-1878). American scientist. He discovered magnetic induction independently. The unit of inductance *henry* is named after him.
6. Ernst Werner von Siemens (1816-1892). German industrialist. Founder of Siemens AG. He invented pointer telegraphy and a moving coil transducer, a forerunner of the loudspeaker. The unit of conductance *siemens* is named after him.
7. Gustav Kirchhoff (1824-1887). German physicist. He formulated *KCL* and *KVL* while he was a student. He also worked on thermal radiation and spectroscopy.
8. James Clerk Maxwell (1831-1879). Scottish mathematician and physicist. He unified the theory of electromagnetism in twenty equations which later were simplified to four by Heaviside using vector notations. Maxwell's equations were described in *A Dynamic Theory of the Electromagnetic Field*, 1865. Mesh equations and node equations were introduced in *A Treatise on Electricity and Magnetism*, 1873.
9. Oliver Heaviside (1850-1925). English engineer. He reformulated Maxwell's equations and invented the Heaviside operational calculus, akin to the Laplace transform. He also invented the coaxial cable and studied skin effects on transmission lines. He was believed to be the first to propose the use of loading coils on transmission lines to increase the useful range of long distance telegraphy.
10. Heinrich Rudolf Hertz (1857-1894). He first demonstrated the existence of electromagnetic waves and developed a receiver to detect them. The receiver consisted of an antenna, coupled inductors, a capacitor and a switch (no electronics). The unit of frequency *hertz* is named after him.
11. Léon Charles Thévenin (1857-1926). French electrical engineer of the Thévenin equivalent circuit fame.
12. Michael Idvorsky Pupin (1858-1935). Serbian/American physicist. He obtained patents on the use of loading coils periodically inserted in transmission lines to increase the useful range of long distance telephony.

13. Charles Proteus Steinmetz (1865-1923). American mathematician and electrical engineer. He used complex exponentials to represent voltages and currents in the sinusoidal steady state, thereby the analysis of AC circuits is simplified to one of algebraic manipulation of complex numbers.
14. George Ashley Campbell (1870-1935). American electrical engineer. He did the mathematical analysis of loading coils for long distance telephony and developed the "wave filter" or "image parameter" theory to aid the design of filters for carrier telephony. The theory is described in *Physical Theory of Electric Wave Filters*, 1922.
15. R. M. Foster. American electrical engineer. He presented for the first time, in 1924, the necessary and sufficient conditions for a rational function to be the impedance of a circuit composed of inductors and capacitors[26].
16. Otto Brune. He was the first to show, in 1931, that any positive real function can be realized as the input impedance of a circuit consisting of resistors, capacitors, inductors and ideal transformers [12].
17. Edward Lawry Norton (1898-1983). American engineer of the Norton equivalent circuit fame, though his principal occupation was filter design.
18. Ernst Adolf Guillemin (1898-1970). Professor of Electrical Engineering, Massachusetts Institute of Technology, known as the father of "modern" circuit theory. He brought what was then higher mathematics (functions of a complex variable, linear algebra, graph theory) to bear on the study of circuits. He published many texts on circuit theory [29, 28] and produced many excellent students who later became outstanding circuit theorists.
19. Bernard D. H. Tellegen (1900-1990). Dutch electrical engineer of the Tellegen's theorem fame. He also invented the pentode in 1926 and the gyrator in 1948.
20. Wilhelm Cauer (1900-1945). German mathematician and electrical engineer. He developed a mathematical theory of circuits that led to systematic design of prototype filters, including those with equal-ripple in the pass and stop bands (elliptic function filters).
21. Hendrik Wade Bode (1905-1982). American physicist and electrical engineer. He derived many of the analytic properties and theoretical limits of circuit functions in a classic book on circuit theory: *Network Analysis and Feedback Amplifier Design*. Van Nostrand, 1945.
22. Sidney Darlington (1906-1997). American physicist and electrical engineer. He was first to show that a positive real function can be realized as an impedance of a lossless two-port terminated in a one-ohm resistor. He developed the theory of "insertion loss" synthesis of a transfer function of a lossless two-port terminated in a one-ohm resistor [19]. The bipolar transistor circuit *Darlington pair* is named after him.
23. R. Bott and R. J. Duffin. They showed in 1948 that any positive real function can be realized as the input impedance of a circuit consisting of resistors, capacitors and inductors, without transformers [9].
24. Mac Elwyn Van Valkenburg (1921-1997). Professor of Electrical Engineering, University of Illinois, Urbanna, and Princeton University. More than anyone else, he promoted the teaching of mathematical theory of circuit analysis and synthesis

throughout the United States. His books on circuit theory were well-known and well-received for their lucidity and informal style [65, 66]. He also produced a large number of distinguished Ph.D. students who later contributed much to the advancement of circuit theory.

The list ends here for now, to be continued in the near future. Contemporary circuit theorists will be recognized in time for their contributions to the development of theory of broadband matching, active filter design, computer-aided design of very-large-scale integrated circuits, nonlinear circuit analysis, and distributed circuit theory.

Chapter 2
Fundamentals

The foundation of circuit theory rests on two physical laws: that the sum of the voltages along a closed path is zero, and that the sum of the currents leaving a node is zero. With these two laws, together with the equations that define the elements, we have a complete mathematical description of all circuits. In this chapter, we derive a fundamental and compact set of differential-algebraic equations which describe the behavior of any circuit at all times. We also prove the all-important Tellegen's theorem, which is fundamental in the derivation of properties of circuits.

2.1 Kirchhoff's laws

A circuit is an assembly of *elements* whose *terminals* are connected at *nodes*. There are basically seven kinds of elements that make up all circuits: the resistor, capacitor, inductor, voltage source, current source, diode, and the transistor. It is remarkable that by a judicious choice of element kinds, element sizes, and circuit topology, we can construct circuits as building blocks of such complex systems as computers, communication transceivers, audio-video entertainment systems, weapon systems, and medical diagnosis systems.

Circuit elements have terminals. Some have two: resistors, capacitors, inductors, voltage and current sources, and diodes. Some have three: bipolar transistors. Some have four: *MOS* (metal oxide semiconductor) transistors; or more: coupled inductors. Between every pair of terminals there exists a voltage (potential drop) and in each terminal there flows a current. The fundamental assumption of circuit theory is that the voltages satisfy Kirchhoff's voltage law (*KVL*) and the currents satisfy Kirchhoff's current law (*KCL*). *KVL* states that the sum of the voltages along a loop is zero and *KCL* states that the sum of the currents meeting at a node is zero.

The physical basis of *KVL* is Ampère's law which states that the line integral of the electric field along a closed path is zero, provided the loop does not enclose any changing magnetic field. *KCL* is based on charge conservation. It follows that the sum of the terminal currents of an element is zero.

Given a circuit, we can write a *KVL* equation for every loop and a *KCL* equation for every node. The set of *KVL* equations for the loops and the set of *KCL* equations for the nodes, together with the equations that define the current-voltage equations of the elements, constitute the circuit equations that describe the state of being of the circuit at all times. The set of terminal voltages and terminal currents that satisfy these three sets of equations is called the *solution* of the circuit. Later, we will see that the circuit equations are a set of differential-algebraic equations. By the theory of such equations, the solution is unique for each set of initial conditions.

It should be noted that the *KVL* equations are *linear homogeneous equations* of the form

$$v_1 + v_2 + \cdots + v_p = 0\,, \tag{2.1}$$

where the voltages form a loop. Similarly, the *KCL* equations are linear homogeneous equations of the form

$$i_1 + v_2 + \cdots + i_q = 0\,, \tag{2.2}$$

where the currents meet at a node or are the terminal currents of an element. In general, the voltages and currents are all functions of time t.

2.2 Linear and nonlinear elements

For pedagogical purposes, circuit elements are best classified as linear and nonlinear elements. An element is *linear* if and only if the terminal voltage v and terminal current i, together with the initial condition, if any, satisfy the homogeneity and additivity properties in the equation that defines the element; otherwise, it is *nonlinear*.Homogeneity says that if $\{v(t), i(t), u(0)\}$ satisfies the element equation, where $u(0)$ is the initial condition, so does $\{Av(t), Ai(t), Au(0)\}$ for any constant A. Additivity says that if $\{v'(t), i'(t), u'(0)\}$ and $\{v''(t), i''(t), u''(0)\}$ separately satisfy the element equation, so does $\{v'(t)+v''(t), i'(t)+i''(t), u'(0)+u''(0)\}$.

It is easy to establish that the resistor is a linear element. Its definition is

$$v = Ri\,, \qquad \text{or} \qquad i = Gv\,, \tag{2.3}$$

where R is the resistance in ohms (Ω) and G the conductance in siemens (S). So is a capacitor, whose definition is

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t}\,, \qquad \text{or} \qquad v(t) = v(0) + \frac{1}{C}\int_0^t i(\tau)\mathrm{d}\tau\,, \tag{2.4}$$

where C is the capacitance in farads (F). The initial capacitor voltage $v(0)$ is part of the definition. The inductor is also linear, as can be seen from its definition:

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}\,, \qquad \text{or} \qquad i(t) = i(0) + \frac{1}{L}\int_0^t v(\tau)\mathrm{d}\tau\,, \tag{2.5}$$

where L is the inductance in henrys (H) and $i(0)$ is the initial current. Similarly, a system of coupled inductors is also linear. The definition is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} L_{11} & M \\ M & L_{22} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{2.6}$$

with some given initial values of the currents.

Next consider an independent voltage source, whose terminal relation is

$$v(t) = e(t)\,, \qquad \text{for all values of } i(t)\,, \tag{2.7}$$

where $e(t)$ is some known function of t. The relation says that $v(t)$ is independent of $i(t)$. So are $Av(t)$ and $v(t) = v'(t) + v''(t)$ for any $v'(t)$ and $v''(t)$, and the element is linear.

Similarly, an independent current source is a linear element. Its definition is

$$i(t) = j(t)\,, \qquad \text{for all values of } v(t)\,, \tag{2.8}$$

where $j(t)$ is some function of t.

All electronic elements are nonlinear. For example, the $I - V$ relation of a *pn*-diode is

$$I = I_s\left(\mathrm{e}^{V/v_t} - 1\right)\,, \tag{2.9}$$

where I_s and v_t are constants. It is clear that $\{V, I\}$ does not satisfy homogeneity nor additivity. Consider an *npn* bipolar transistor operating at low frequencies. Its current-voltage relation is given by

$$I_c = \alpha I_{be}\left(\mathrm{e}^{V_{be}/v_t} - 1\right) - I_{bc}\left(\mathrm{e}^{V_{bc}/v_t} - 1\right)\,, \tag{2.10}$$

$$I_e = I_{be}\left(\mathrm{e}^{V_{be}/v_t} - 1\right) - \alpha_r I_{bc}\left(\mathrm{e}^{V_{bc}/v_t} - 1\right)\,, \tag{2.11}$$

$$I_b = I_e - I_c\,, \tag{2.12}$$

where the collector current I_c, emitter current I_e and base current I_b are all nonlinear functions of the base-emitter voltage V_{be} and base-collector voltage V_{bc}, and where α, α_r, I_{be}, I_{bc} and v_t are constants. In these equations, neither homogeneity nor additivity holds.

Lastly, in a much simplified model of a four-terminal *nMOS* transistor, the drain-source current I_{ds} is a function of the three terminal-pair voltages: drain-source voltage V_{ds}, gate-source voltage V_{gs}, and source-bulk voltage V_{sb}, as follows:

$$I_{ds} = K_1[(V_{gs} - V_T)V_{ds} - K_2 V_{ds}^2]\,, \quad \text{with} \tag{2.13}$$

$$V_T = K_3 + K_4\sqrt{V_{sb} + K_5}\,, \tag{2.14}$$

where $K_1, \ldots, K_5$ are constants. The gate current and bulk current are assumed to be zero. We see the *MOS* transistor is a nonlinear element.

2.3 Linear and nonlinear circuits

Let the solution of a circuit be $\{v(t), i(t), u(0)\}$, where $\{v(t)\}$ is the set of terminal voltages of the elements, $\{i(t)\}$ the set of terminal currents, and $\{u(0)\}$ the set of initial conditions. By definition, $\{v(t)\}$ satisfies the *KVL* equations, $\{i(t)\}$ satisfies the *KCL* equations, and $\{v(t), i(t), u(0)\}$ satisfies the definitions of the elements. We say that the circuit is *linear* if and only if $\{v(t), i(t), u(0)\}$ satisfies the homogeneity and additivity properties. That is, $\{Av(t), Ai(t), Au(0)\}$ is a solution of the circuit for any constant A, and if $\{v'(t), i'(t), u'(0)\}$ and $\{v''(t), i''(t), u''(0)\}$ are distinct solutions, so is $\{v'(t)+v''(t), i'(t)+i''(t), u'(0)+u''(0)\}$; otherwise, the circuit is *nonlinear*.

Theorem 2.1. *A circuit composed entirely of linear elements is a linear circuit.*

Proof. Let $\{v(t), i(t), u(0)\}$ be the solution of the circuit. Since the elements are linear by hypothesis, in the equations that define the elements, homogeneity and additivity hold for $\{v(t), i(t), u(0)\}$. Since *KVL* and *KCL* equations are homogeneous, $\{Av(t)\}$ and $\{Ai(t)\}$ satisfy the *KVL* equations and *KCL* equations, respectively. Since *KVL* and *KCL* are linear equations, additivity holds for $\{v(t)\}$ and for $\{i(t)\}$ in these equations. Therefore $\{v(t), i(t), u(0)\}$ satisfies the homogeneity and additivity properties and by definition the circuit is linear. $\square$

The theorem says that it is not possible to construct a nonlinear circuit out of linear elements. One can never make a multiplier, for example, using only resistors, capacitors and inductors.

Theorem 2.2 (Superposition theorem). *Let $\mathscr{N}$ be a linear circuit without internal sources. 1. Homogeneity: Let $e(t)$ be a voltage source inserted into $\mathscr{N}$. Let $v_p(t)$ be the terminal voltage across some element, regarded as the response to the excitation $e(t)$. Then $Av_p(t)$ is the response to $Ae(t)$ for any constant A. 2. Additivity: Let $v_p^{(1)}(t)$ be the response, taken across some element, to an excitation $e_1(t)$ inserted at some point in $\mathscr{N}$, and let $v_p^{(2)}(t)$ be the response, taken across the same element, to an excitation $e_2(t)$ inserted at another point. Then $v_p^{(1)}+v_p^{(2)}$ is the response to $e_1(t)$ and $e_2(t)$ applied simultaneously.*

This is a well-known and important theorem about linear circuits. The proof is straightforward and will be omitted. It is obvious that the excitation can also be a current source connected across a terminal pair and the response can be the current in any terminal.

In general, superposition does not hold in a nonlinear circuit. It follows that such useful techniques as Laplace transform and Fourier analysis are not applicable in nonlinear circuits.

From this point on, throughout this book, we shall study only linear circuits, namely circuits composed of resistors, capacitors, inductors, and coupled inductors. Unless otherwise stated, we regard all independent voltage and current sources as sources of external excitations. We call such circuits *RLC circuits* for short. Nonlinear circuits in general require different analysis and design methods that are best taken up in a separate volume. In fact, nonlinear circuits are usually subdivided into digital electronic circuits as components of a computer or a digital signal processing system, and analog electronic circuits as components of a communication system. Indeed, it is the nonlinearity of the electronic devices that makes it possible to design such circuits as logic gates, amplifiers, multipliers, modulators, and oscillators. These circuit classes are best studied individually.

2.4 Small-signal equivalent circuits

There is a class of "linear" circuits that is noteworthy. These are small-signal equivalent circuits *derived* from electronic circuits. They are artificial circuits invented to aid the analysis and design of such circuits as active filters and operational amplifiers. In an electronic circuit, the devices are usually biased with a DC voltage or current. Superimposed on the bias voltage and current are signals whose amplitude is much smaller, by about three orders of magnitude, than the bias. To fix idea, consider an *MOS* transistor. Let the drain-source, gate-source, and source-bulk voltages, and drain current be, respectively

$$V_{ds} = \overline{V}_{ds} + v_{ds}(t)\,, \tag{2.15}$$
$$V_{gs} = \overline{V}_{gs} + v_{gs}(t)\,, \tag{2.16}$$
$$V_{sb} = \overline{V}_{sb} + v_{sb}(t)\,, \tag{2.17}$$
$$I_{ds} = \overline{I}_{ds} + i_{ds}(t)\,, \tag{2.18}$$

where the overbarred quantities denote the DC bias and the lower-case quantities the small signal voltages and current. From Eq.(2.13), we expand the drain current I_{ds} about the bias. Keeping only DC and first order terms, we get

$$\begin{aligned} I_{ds} &= I_{ds}(\overline{V}_{ds}, \overline{V}_{gs}, \overline{V}_{sb}) + \left.\frac{\partial I_{ds}}{\partial V_{ds}}\right|_{V_o} v_{ds}(t) + \left.\frac{\partial I_{ds}}{\partial V_{gs}}\right|_{V_o} v_{gs}(t) + \left.\frac{\partial I_{ds}}{\partial V_{sb}}\right|_{V_o} v_{sb}(t)\,, \\ &= \overline{I}_{ds} + i_{ds}(t)\,, \end{aligned} \tag{2.19}$$

where the partial derivatives are evaluated at the DC bias $V_o = \{\overline{V}_{ds}, \overline{V}_{gs}, \overline{V}_{sb}\}$. Equating the DC and time-varying small-signal terms separately, we get

$$\overline{I}_{ds} = I_{ds}(\overline{V}_{ds}, \overline{V}_{gs}, \overline{V}_{sb})\,, \tag{2.20}$$
$$i_{ds}(t) = g_d\, v_{ds}(t) + g_m\, v_{gs}(t) + g_b\, v_{sb}(t)\,, \tag{2.21}$$

where

$$g_d = \left.\frac{\partial I_{ds}}{\partial V_{ds}}\right|_{V_o}, \tag{2.22}$$

$$g_m = \left.\frac{\partial I_{ds}}{\partial V_{gs}}\right|_{V_o}, \tag{2.23}$$

$$g_b = \left.\frac{\partial I_{ds}}{\partial V_{sb}}\right|_{V_o}, \tag{2.24}$$

and are called, respectively, the *drain conductance*, *gate transconductance*, and *bulk transconductance* of the transistor. So insofar as the small signals are concerned, the transistor is equivalent to a two-terminal element consisting of a conductance in parallel with two current sources whose values are controlled by the gate-source voltage and source-bulk voltage, respectively. The important thing to note is that this equivalent element is *linear*. If we replace each and every transistor in a circuit by its small-signal equivalent, we will have created a circuit with the same topology as the original but it now contains only linear elements and therefore the new circuit is linear. We call the new circuit *small-signal equivalent circuit*. In the equivalent circuit, the circuit variables are the small-signal voltages and currents, and they satisfy *KVL*, *KCL*, and the linearized element equations. Separately the bias voltages and currents satisfy *KVL*, *KCL*, and the nonlinear element equations.

It must be emphasized that the small-signal equivalent circuit does not exist physically by itself. It is derived from and supported by the nonlinear circuit which consists of the transistors. It is an artificial entity which is used to describe the small-signal components of the signals in the circuit. For this purpose, we have introduced a new linear element, a current source whose value is controlled by a voltage not across the terminals of the element but across terminals different from them. We call such an element a *voltage-controlled current source*.

The presence of voltage-controlled current sources materially changes some of the properties of *RLC* circuits. We shall point out the differences from time to time.

2.5 Fundamental *KVL* equations

We now turn to circuit analysis. We need a mathematical description of a circuit that is easy to formulate, economical in the number of equations needed, and applicable to circuits of any topology. We will show that in any circuit, there exists a maximally independent set of *KVL* equations and a maximally independent set of *KCL* equations. When these two sets of equations are combined with the equations that define the elements, we obtain a compact set of equations commonly known as *modified node equations*, or *hybrid equations* or just *node equations* for short.[1] Once we have the solution of these equations, the solution of the circuit can be deduced from it.

[1] We will use these names interchangeably.

2.5.1 Conventions

Consider an *RLC* circuit. For each element, we define a *terminal voltage* as the voltage (potential drop) that exists across its two terminals, and we define a *terminal current* as the current that enters the element in one terminal and leaves the other. By convention, the terminal voltage and terminal current are assigned the same direction. Fig. 2.1 shows the assignment of terminal voltages and terminal currents of a circuit of seven elements and five nodes.

Let the elements of a circuit be connected at n nodes. We choose a node to be the *ground node* and assign a voltage to each of the non-ground nodes with respect to ground and call the $n-1$ such voltages the *node voltages* of the circuit. Fig. 2.1 shows how the node voltages $v_{1n}, v_{2n}, v_{3n}, v_{4n}$ are defined.

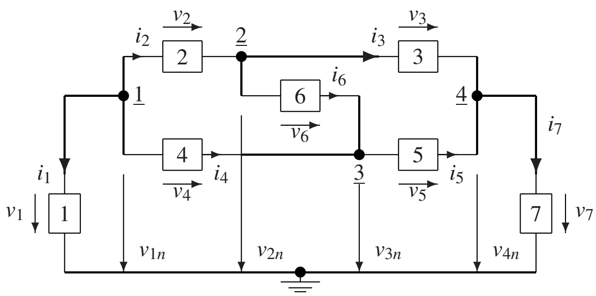

Fig. 2.1 Assignment of voltages and currents in a circuit

2.5.2 KVL *equations*

We first note that every element is connected to two nodes. As evident in Fig. 2.1, each terminal voltage together with the two node voltages of the nodes to which the element is connected forms a loop. Writing a *KVL* equation for each such loop, we can express the terminal voltages in terms of the $n-1$ node voltages uniquely, as follows.

$$v_1 = v_{1n}\,, \tag{2.25}$$
$$v_2 = v_{1n} - v_{2n}\,, \tag{2.26}$$
$$v_3 = v_{2n} - v_{4n}\,, \tag{2.27}$$
$$v_4 = v_{1n} - v_{3n}\,, \tag{2.28}$$
$$v_5 = v_{3n} - v_{4n}\,, \tag{2.29}$$
$$v_6 = v_{2n} - v_{3n}\,, \tag{2.30}$$
$$v_7 = v_{4n}\,. \tag{2.31}$$

Transposing the right-hand-side of the equations to the left, we have in matrix form:

$$\begin{bmatrix}\mathbf{U}\ \mathbf{A}\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{v_n}\end{bmatrix} = 0\,, \tag{2.32}$$

where $\mathbf{v}$ is the vector of terminal voltages, $\mathbf{v_n}$ the vector of node voltages, $\mathbf{U}$ a unit matrix, and $\mathbf{A}$ a matrix of -1, 0, and 1. Equations (2.25 - 2.31) are all the *KVL* equations we need to describe the voltages of the circuit completely, because any other *KVL* equation can be expressed as a linear combination of these equations. For example, the *KVL* equation

$$-v_1 + v_2 + v_6 + v_5 + v_7 = 0 \tag{2.33}$$

is obtained from the following combination:

$$-\mathrm{Eq}(2.25) + \mathrm{Eq}(2.26) + \mathrm{Eq}(2.30) + \mathrm{Eq}(2.29) + \mathrm{Eq}(2.31) = 0\,, \tag{2.34}$$

and so forth. We can state these observations in a theorem:

Theorem 2.3 (KVL). *Let there be b terminal voltages in a circuit of n nodes. Then the set of b* KVL *equations, each expressing a terminal voltage in terms of the $n-1$ node voltages, is a maximally independent set of* KVL *equations.*

Proof. That the equations are linearly independent is evident from the fact that in each equation there is one variable, the terminal voltage for which the *KVL* equation is written, which appears in no other *KVL* equation. To prove the set is maximally independent, we assume the contrary. Suppose there exists another *KVL* equation which is linearly independent of the set. The loop of this *KVL* equation must contain at least two terminal voltages. Suppose there are exactly two, v_{e1} and v_{e2}. These terminal voltages are necessarily adjacent, for if not, they cannot be in the same loop. Let the nodes of v_{e1} be node 1 and node 2. Let the nodes of v_{e2} be node 2 (why?) and node 3. Then the *KVL* equation of this loop is

$$-v_{n1} + v_{e1} + v_{e2} + v_{n3} = 0\,. \tag{2.35}$$

But it is expressible as

$$-v_{n1} + v_{e1} + v_{n2} - v_{n2} + v_{e2} + v_{n3} = 0\,. \tag{2.36}$$

The first three terms appear in the *KVL* equation of v_{e1} and the last three in the *KVL* equation of v_{e2}. So the *KVL* equation (2.35) is a linear combination the *KVL* equations written for v_{e1} and v_{e2}, contrary to the assumption that it is linearly independent. It follows there cannot be any other independent *KVL* equation and the set of b *KVL* equations written for the b terminal voltages are maximally independent. □

The set of b *KVL* equations expressing each terminal voltage in terms of the $n-1$ node voltages will be called the *fundamental KVL equations* of the circuit. The significance of Theorem 2.3 is that to describe the voltages in a circuit, we need define only the $n-1$ node voltages. All the terminal voltages can be expressed in terms of the node voltages uniquely. In practical terms, we are saying that in the laboratory or in the field, we need only measure the voltages at the nodes of a circuit (with respect to ground). Once these are known, all the other voltages in the circuit are determined.

2.6 Fundamental *KCL* equations

With reference to Fig. 2.1, for each node, we can write a *KCL* equation. Using the convention that a current leaving a node is assigned a positive sign, we obtain the following five *KCL* equations for the circuit:

$$i_1 + i_2 + i_4 = 0, \tag{2.37}$$
$$-i_2 + i_3 + i_6 = 0, \tag{2.38}$$
$$-i_3 - i_5 + i_7 = 0, \tag{2.39}$$
$$-i_4 + i_5 - i_6 = 0, \tag{2.40}$$
$$-i_1 - i_7 = 0. \tag{2.41}$$

We observe that in the set of *KCL* equations, each current variable appears once with a positive sign and once with a negative sign. This follows since each terminal current must leave a node and enter another. So the sum of all *KCL* equations is identically zero. Any equation, in particular the one written for the ground node, is redundant and need not be written. Therefore, to describe the currents in a circuit, we only need to write $n-1$ *KCL* equations, one for each of the non-ground nodes. Moreover, we have, similarly to *KVL*,

Theorem 2.4 (KCL). *In a circuit of n nodes without disconnected parts, the set of* KCL *equations written for the* $n-1$ *non-ground nodes is maximally independent.*

The proof is left to the reader as an exercise. The set of *KCL* equations written for the $n-1$ non-ground nodes will be called the *fundamental KCL equations* of the circuit.

2.7 Tellegen's theorem

A direct consequence of *KVL* and *KCL* is Tellegen's theorem [63], which is stated as follows.

Theorem 2.5 (Tellegen's theorem). *Let N_1 and N_2 be two topologically identical circuits, namely, they have the same number of nodes and elements, and their elements are connected in the same way. Let $\mathbf{v}^{(j)}$ be the vector of terminal voltages and $\mathbf{i}^{(j)}$ be the vector of terminal currents of N_j, $j = 1, 2$. Then*

$$[\mathbf{v}^{(j)}]^T [\mathbf{i}^{(k)}] = 0\,, \quad \textit{for} \quad j,k \in \{1,2\}\,. \tag{2.42}$$

Before we prove the theorem, let us apply it to the circuit of Fig. 2.1. The theorem says that

$$v_1 i_1 + v_2 i_2 + v_3 i_3 + v_4 i_4 + v_5 i_5 + v_6 i_6 + v_7 i_7 = 0\,. \tag{2.43}$$

Moreover, if we have two different circuits with the same topology, then the sum of the voltage-current products, with the voltages taken from one circuit and the currents taken from the other circuit, is zero.

By expanding Eq. (2.43), we get a clue as to how we might prove the theorem. Expressing each terminal voltage in terms of the node voltages, we have from Sect. 2.5.2:

$$\begin{aligned} v_{1n} i_1 + (v_{1n} - v_{2n}) i_2 + (v_{2n} - v_{4n}) i_3 + (v_{1n} - v_{3n}) i_4 \\ + (v_{3n} - v_{4n}) i_5 + (v_{2n} - v_{3n}) i_6 + v_{4n} i_7 = \\ v_{1n}(i_1 + i_2 + i_4) + v_{2n}(-i_2 + i_3 + i_6) + v_{3n}(-i_4 + i_5 - i_6) \\ + v_{4n}(-i_3 - i_5 + i_7) = 0\,. \end{aligned}$$

The coefficient on each node voltage variable is zero because it is the sum of the currents at the node of the node voltage.

Proof. Let v_k be the terminal voltage of some element connected between node p and node q. Then we have

$$v_k = v_{pn} - v_{qn}\,. \tag{2.44}$$

By convention, terminal current i_k leaves node p and enters node q. The term $v_k i_k$ becomes

$$v_k i_k = v_{pn} i_k + v_{qn}(-i_k)\,. \tag{2.45}$$

We repeat the expansion for all terminal voltages (all k), and we get

$$\sum_k v_k i_k = v_{1n}\Big(\sum_{j_1} i_{j_1}\Big) + v_{2n}\Big(\sum_{j_2} i_{j_2}\Big) + \cdots + v_{(n-1)n}\Big(\sum_{j_{(n-1)}} i_{j_{(n-1)}}\Big)\,, \tag{2.46}$$

where i_{j_m} are the currents that leave or enter node m with their signs being positive or negative, respectively. By *KCL*, each such sum is zero, and we have proved Eq. (2.42) for the case $j = k$.

If we have two topologically identical circuits, the terminal voltages of the first circuit are expressed in terms of the node voltages as before. The terminal currents of the second will appear at each node in the same way as the terminal currents of the first circuit because the elements of the second circuit are connected in the same way as the first. So in the expansion of the sum of voltage-current products, the current terms associated with a node voltage will appear in the same way as the case of a single circuit. Since the sum of currents at a node is zero, the sum of voltage-current products in Eq. (2.42) is again zero for $j \neq k$. □

In words, the theorem says that if $j = k$, the total power taken by the elements of a circuit is zero. If $j \neq k$, the theorem says that the sum of products of terminal voltages in N_1 and the corresponding terminal currents in N_2 is zero.

One of the remarkable features of Tellegen's theorem is that it holds true regardless of the kinds of elements each circuit may contain. The elements may be linear or nonlinear. The elements in one circuit may be different in kind from the corresponding elements in the other. As long as the topologies of the two circuits are the same, the sum of the products of the voltages in one and the currents in the corresponding elements in the other is zero.

Moreover, since terminal voltages obey *KVL* and terminal currents obey *KCL* at all times, the voltages in one circuit may be taken at time t_1 and the currents in the other circuit taken at time $t_2 \neq t_1$, and the sum of the products is zero. We will have occasions to verify Tellegen's theorem with numerical examples shortly. The theorem can be extended to include circuits composed of multi-terminal elements such as transistors. (See Problem 2.4.)

2.8 Energy in coupled inductors

Consider a system of coupled inductors. Let $\mathbf{v}$ be the vector of terminal voltages and $\mathbf{i}$ the vector of terminal currents. Then by definition we have

$$\mathbf{v} = L\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t}, \tag{2.47}$$

where L is the inductance matrix.

Theorem 2.6. *The inductance matrix L of a system of coupled inductors is symmetric, i.e., $L = L^T$.*

Proof. The power taken by the system is

$$\mathbf{i}^{\mathbf{T}}\mathbf{v} = \sum_m \sum_n L_{mn} i_m \frac{\mathrm{d}i_n}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}t}, \tag{2.48}$$

where w is the energy. The incremental energy increase due to incremental increases in the currents is

$$\mathrm{d}w = \sum_m \sum_n L_{mn} i_m \mathrm{d}i_n \,. \tag{2.49}$$

Since w is a function of the currents, the incremental energy is

$$\mathrm{d}w = \sum_n \frac{\partial w}{\partial i_n} \mathrm{d}i_n \,. \tag{2.50}$$

Noting the currents are independent variables, we get

$$\frac{\partial w}{\partial i_n} = \sum_m L_{mn} i_m \,, \tag{2.51}$$

$$\text{and} \qquad \frac{\partial^2 w}{\partial i_m \partial i_n} = L_{mn} \,. \tag{2.52}$$

It follows that $L_{mn} = L_{nm}$ and $L = L^T$. □

The theorem says that the coupling from inductor A to inductor B is the same as the coupling from B to A.

Let us re-write the power $p(t)$ as follows.

$$p(t) = \mathbf{i}^{\mathbf{T}} L \frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = p^T(t) = \frac{\mathrm{d}\mathbf{i}^T}{\mathrm{d}t} L \mathbf{i} \,, \tag{2.53}$$

and

$$p(t) + p^T(t) = 2p(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbf{i}^T L \mathbf{i} \right) \,. \tag{2.54}$$

Integrating, we find the total energy to be

$$w(t) = \frac{1}{2} \mathbf{i}^T(t) L \mathbf{i}(t) \,, \tag{2.55}$$

which is a quadratic form. Since the energy stored in the inductors must be non-negative for all values of the currents, the constant of integration is zero and we have

Theorem 2.7. *The inductance matrix of a system of coupled inductors is positive semi-definite.*

A consequence of this theorem is that in a system of two coupled inductors, the coefficient of coupling is less than or equal to one, a well-known physical fact.

2.9 Passive circuits

Consider an *RLC* circuit with an external voltage source $e(t)$ connected across some terminals. Let its current be $i(t)$. By Tellegen's theorem, the power delivered to the circuit equals the sum of the powers taken by the elements of the circuit. Substituting the definitions of the elements and collecting terms, we get

$$e(t)i(t) = \sum_k R_k i_{R_k}^2 + \sum_k L_k i_{L_k} \frac{\mathrm{d}i_{L_k}}{\mathrm{d}t} + \sum_k C_k v_{C_k} \frac{\mathrm{d}v_{C_k}}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{i}_m^T L \mathbf{i}_m\right), \tag{2.56}$$

where R_k are the resistances of the resistors and i_{R_k} the resistor currents; L_k the inductances and i_{L_k} the inductor currents; C_k are the capacitances and v_{C_k} the capacitor voltages; lastly L is the inductance matrix of the coupled inductors and $\mathbf{i}_m$ is the vector of currents in these inductors. The total energy taken by the circuit is

$$w(t) = \sum_k R_k \int_{-\infty}^{t} i_{R_k}^2(u)\,\mathrm{d}u + \frac{1}{2}\sum_k L_k i_{L_k}^2(t) + \frac{1}{2}\sum_k C_k v_{C_k}^2(t) + \frac{1}{2}\mathbf{i}_m^T(t) L \mathbf{i}_m(t). \tag{2.57}$$

The last term is non-negative as noted in Theorem 2.7. The first three terms are non-negative for all values of the currents and voltages if the element values are non-negative, which is the case in an *RLC* circuit. It follows that the energy taken by an *RLC* circuit is non-negative for all t.

We define a *passive circuit* as one in which the energy delivered to it by an external source is non-negative for all t, and we have

Theorem 2.8. *All* RLC *circuits are passive.*

In words, a passive circuit only takes energy from an external source. It does not return any energy to the source. If it does, the circuit must have an internal source of energy and it is called an *active circuit*.

All electronic circuits except diode circuits are active circuits. Most operate with signal amplitudes comparable to the DC biases, for example, digital circuits and communication circuits used in transceivers. In a small-signal equivalent circuit, the bias sources are omitted but physically they are part of the circuit. For this reason, we refer to small-signal equivalent circuits as *small-signal active circuits*.

It is possible to create a negative resistance in a small-signal active circuit. For example, take two identical *MOS* transistors A and B. Cross-couple them such that the drain of A is connected to the gate of B, and vice versa. Let the source terminals of the transistors be connected to a common current source. Assume the drain conductance is negligible. Then the input resistance looking into the two drain terminals is $-2/g_m$, where g_m is the gate transconductance.

2.10 Modified node equations

We now take up the formulation of circuit equations whose solution yields the circuit voltages and currents that satisfy *KVL*, *KCL*, and the definitions of the elements. As mentioned earlier, we look for a simple way to write a compact set of equations that is applicable to all circuits. Consider the circuit of Fig. 2.2, which has the same topology as the circuit of Fig. 2.1. We begin with the fundamental *KVL* equations, repeated here:

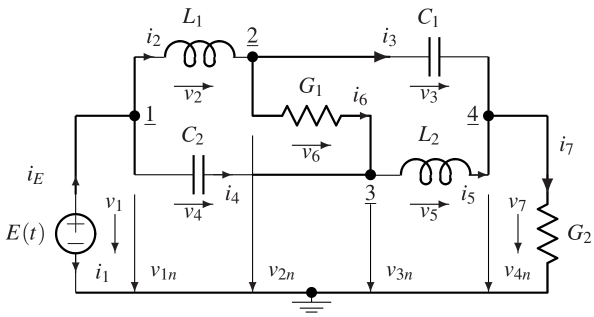

Fig. 2.2 A circuit to illustrate formulation of node equations

$$
\begin{aligned}
v_1 &= v_{1n} & v_2 \,, &= v_{1n} - v_{2n} \,, \\
v_3 &= v_{2n} - v_{4n} \,, & v_4 &= v_{1n} - v_{3n} \,, \\
v_5 &= v_{3n} - v_{4n} \,, & v_6 &= v_{2n} - v_{3n} \,, \\
v_7 &= v_{4n} \,,
\end{aligned}
$$

together with the fundamental *KCL* equations:

$$
\begin{aligned}
i_1 + i_2 + i_4 &= 0 \,, \\
-i_2 + i_3 + i_6 &= 0 \,, \\
-i_4 - i_6 + i_5 &= 0 \,, \\
-i_3 - i_5 + i_7 &= 0 \,.
\end{aligned}
$$

The voltages and currents are related by the element equations:

$$
\begin{aligned}
v_1 &= E(t) \,, & v_2 &= L_1 \frac{\mathrm{d}i_2}{\mathrm{d}t} \,, \\
i_3 &= C_1 \frac{\mathrm{d}v_3}{\mathrm{d}t} \,, & i_4 &= C_2 \frac{\mathrm{d}v_4}{\mathrm{d}t} \,, \\
v_5 &= L_2 \frac{\mathrm{d}i_5}{\mathrm{d}t} \,, & i_6 &= G_1 v_6 \,, \\
i_7 &= G_2 v_7 \,.
\end{aligned}
$$

Now using *KVL*, we express each terminal voltage in terms of the node voltages. Substitute the element equations into the *KCL* equations and append the equations for the inductors and voltage sources to the *KCL* equations, while retaining as circuit variables the node voltages, currents in the inductors, and the current in the voltage source. For emphasis, we have replaced the inductor current i_2 with i_{L1}, i_5 with i_{L2}, and the current i_1 with $-i_E$. After collecting terms, we obtain the following equations.

$$\begin{bmatrix} C_2 & 0 & -C_2 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & -C_1 & 0 & 0 & 0 \\ -C_2 & 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & -C_1 & 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v_{1n} \\ v_{2n} \\ v_{3n} \\ v_{4n} \\ i_{L1} \\ i_{L2} \\ i_E \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & G_1 & -G_1 & 0 & -1 & 0 & 0 \\ 0 & -G_1 & G_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & G_2 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{1n} \\ v_{2n} \\ v_{3n} \\ v_{4n} \\ i_{L1} \\ i_{L2} \\ i_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E(t) \end{bmatrix} . \tag{2.58}$$

These equations are called the *modified node equations* of the circuit. The first $n-1$ equations are *KCL* equations for the capacitors and resistors. The next set of equations are *KVL* equations for the inductors and voltage sources, all written in terms of the node voltages, inductor currents and voltage source currents. The modified node equations are of the form:

$$M\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + N\mathbf{x} = u(t)\,, \tag{2.59}$$

where $\mathbf{x}$ is the vector of node voltages, inductor currents, and voltage source currents. M and N are constant matrices, and $\mathbf{u}(\mathbf{t})$ is a vector of sources. Starting with a zero matrix, we construct $M = [m_{ij}]$ as follows.

1. If a capacitor C is connected between node r and node s, add C to m_{rr} and m_{ss}, and add $-C$ to m_{rs} and m_{sr}. Omit the last three if the capacitor is connected between node r and ground.
2. If x_s is assigned to the inductor current of an inductor L, add L to m_{ss}.
3. For a pair of coupled inductors, if x_s is assigned to the current of the primary inductor and x_r to the current of the secondary inductor, then we add L_1 to m_{ss}, L_2 to m_{rr}, and L_{12} to m_{sr} and m_{rs}, where L_1 is the primary inductance, L_2 the secondary inductance, and L_{12} the mutual inductance. Similarly for a system of more than two coupled inductors.

We see that M contains a capacitance matrix and an inductance matrix as submatrices, the rest being all zero. The matrix N is of the form:

$$N = \begin{bmatrix} G & D \\ D' & 0 \end{bmatrix}, \tag{2.60}$$

where $G = [g_{ij}]$ is the conductance matrix. Starting with a zero matrix, it is constructed as follows. If a conductance g is connected between node r and node s, add

g to g_{rr} and g_{ss}, and add $-g$ to g_{rs} and g_{sr}. Omit the last three if the conductance is connected between node r and ground.

Matrices $D = [d_{ij}]$ and $D' = [d'_{ij}]$ describe how the inductors and voltage sources are connected to the rest of the circuit. Assume x_s is assigned to an inductor current i_L. If i_L leaves node j and enters node k, then $d_{js} = 1$ and $d_{ks} = -1$. At the same time, the inductor voltage equation appears as

$$L\frac{\mathrm{d}i_L}{\mathrm{d}t} - v_{jn} + v_{kn} = 0\,. \tag{2.61}$$

So $d'_{sj} = -1$ and $d'_{sk} = 1$. Similarly for each of the inductors in a system of coupled inductors. If follows that $D' = -D^T$, the negative transpose of D.

Similar considerations apply to a voltage source. Lastly, if a current source $J(t)$ is connected between node r and node s, then add $J(t)$ to u_r and $-J(t)$ to u_s, for $r,s \in \{1,\ldots,n_1\}$. If x_s is assigned to the current of a voltage source $E(t)$, then $u_s = E(t)$.

If should be appreciated that the construction of the node equations is systematic and can be easily programmed. In fact, most circuit simulation programs formulate the circuit equations in this way.

The modified node equations then are a system of differential-algebraic equations (the last equation being algebraic), or *DAE*. In general, the matrix M may be singular, so it is not always possible to express the equations in derivative-explicit form. There is no general solution to such a system. (See U. M. Ascher and L. R. Petzold [2].) For circuit simulation, we resort to numerical solution, to be discussed next.

2.11 Numerical solution

Our starting point is Eq. (2.59). Let $\mathbf{x}^*(t)$ be the solution over the time interval $[0, T_{max}]$. Beginning with the initial value $\mathbf{x}^*(0) = \mathbf{x}_0$, we generate a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ such that $\mathbf{x}_n$ is a good approximation of $\mathbf{x}^*(t_n)$, where t_n are discrete time points over the interval. Let n_{max} be the number of time points and let

$$h = \frac{T_{max}}{n_{max}}\,. \tag{2.62}$$

With $t_0 = 0$, we take time points such that $t_n - t_{n-1} = h$ for $n = 1,2,\ldots,n_{max}$. We call h the (uniform) *step size*.

To generate the solution sequence, we approximate the derivative $\mathrm{d}\mathbf{x}/\mathrm{d}t$ at time point t_n by some function of the current value $\mathbf{x}_n$ and past values $\mathbf{x}_{n-p}, p = 1,2,\ldots$, and possibly the past derivatives. It is beyond the scope of this book to give a complete exposition on this subject. We will present only the simplest and most reliable algorithm, which is the Backward Euler method.

2.11.1 Backward Euler method

The derivative at time t_n will be approximated by the slope of a straight line passing through the unknown $\mathbf{x}_n$ and its last known value $\mathbf{x}_{n-1}$:

$$\left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{t_n} \approx \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{h}\,. \tag{2.63}$$

Substituting into Eq. (2.59), we obtain an algorithm by which $\mathbf{x}_1, \mathbf{x}_2, \ldots$ are generated:

$$\left[\frac{1}{h}M + N\right]\mathbf{x}_n = \mathbf{u}(t_n) + \frac{1}{h}M\mathbf{x}_{n-1}\,, \qquad n = 1, 2, \ldots, n_{max}\,, \tag{2.64}$$

assuming the excitation $\mathbf{u}(t_n)$ is known or can be computed for all t_n. Formula (2.63) is known as the *Backward Euler method*. It gives a first order approximation of the true solution, meaning the local truncation error $\mathbf{x}^*(t_n) - \mathbf{x}_n$ is proportional to h^2 times the second derivative of $\mathbf{x}^*(\tau)$, where $t_{n-1} < \tau < t_n$, provided $\mathbf{x}(t)$ is "smooth." Moreover, the formula is known to be "stable" in that for a fixed h, as $n \to \infty$ the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ does not diverge.

By algorithm (2.64) we have reduced the solution of a system of *DAE* to the solution of a system of linear algebraic equations, which is a much easier task.[2]

2.11.2 Consistent initial conditions

To proceed with algorithm (2.64), we need to have the initial values of the unknowns. Presumably, the initial values of the capacitor voltages and inductor currents are known or can be computed from the circuit at $t = 0^-$. The initial values of the other circuit variables cannot be arbitrary and must be "consistent," meaning they must satisfy the node equations (*KCL* and *KVL*). To this end, we need to find the solution to the circuit at $t = 0^+$ with each capacitor replaced by a voltage source of value equal to the initial capacitor voltage, and each inductor replaced by a current source of value equal to the initial inductor current. The resultant "initial-value" circuit contains only resistors and sources and can be solved by any method.

Suppose the elements of the circuit of Fig. 2.2 have the following values:

$$\begin{aligned} &G_1 = 0.1\,\mathsf{S}, \qquad &&G_2 = 1\,\mathsf{S}, \qquad &&C_1 = 1\,\mathsf{F}, \\ &C_2 = 2\,\mathsf{F}, &&L_1 = 4\,\mathsf{H}, &&L_2 = 2\,\mathsf{H}, \end{aligned} \tag{2.65}$$

and assume the excitation is given as

$$E(t) = \begin{cases} 1, & \text{for } t \le 0; \\ 1 + 0.2\sin\omega_1 t + \sin\omega_0 t + 0.2\sin\omega_2 t, & \text{for } t > 0. \end{cases} \tag{2.66}$$

[2] In MATLAB®, the linear equation $Ax = b$ is solved by invoking $x = A\backslash b$.

It is a sum of a DC term, a low-frequency, mid-frequency, and high-frequency sinusoidal terms, the frequencies being $\omega_1 = 1/20$, $\omega_0 = 1/2$, and $\omega_2 = 5$, respectively.

A consistent set of initial values of the node voltages, inductor currents, and voltage source current are:

$$\begin{aligned} &v_{1n}(0) = 1\,\mathsf{V}, \quad &&v_{2n}(0) = 1\,\mathsf{V}, \quad &&v_{3n}(0) = 1/11\,\mathsf{V}, \quad &&v_{4n}(0) = 1/11\,\mathsf{V}, \\ &i_{L1}(0) = 1/11\,\mathsf{A}, \quad &&i_{L2}(0) = 1/11\,\mathsf{A}, \quad &&i_E(0) = 1/11\,\mathsf{A}. \end{aligned} \tag{2.67}$$

The responses are computed over $[0, 300sec]$ from a MATLAB®implementation of

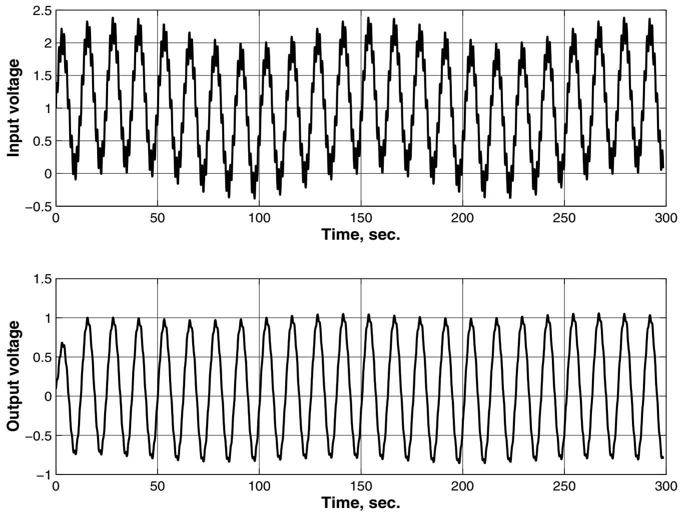

Fig. 2.3 Time response of the circuit of Fig. 2.2 to a sum of DC and three sinusoids. The circuit filters out the low and high frequency components

algorithm (2.64). The time step is chosen so that there are 20 points in a period of the high frequency component. Figure. 2.3 shows the input and output waveforms. The slow amplitude modulation in the input is due to the low frequency component, while the jaggedness is produced by the high frequency component. In the output waveform, we see both components are largely suppressed. Thus this circuit has the properties of a bandpass filter, though not a practical one.

2.11.3 Verification of Tellegen's theorem

It is instructive to verify Tellegen's theorem. We compute the Tellegen sum at time point 100 and it is found to be of the order 10^{-16}, which is essentially zero. We also compute the Tellegen sum at time points 100 and 1000. The sum of products of voltages taken at 1000 and currents taken at 100, and vice versa, is also of the same order. Thus Tellegen's theorem is verified for this circuit.

2.11.4 Remarks

Modified node equations are easy to formulate. The matrices of the equations can be constructed directly from the topology of the circuit. Moreover, the form of the equation is suitable for numerical solution. For these reasons, most circuit simulation programs describe the circuit with modified node equations.

On the other hand, modified node equations do not tell us much about the general behavior of a circuit. It is not apparent how one can deduce properties of a circuit from these equations. For this purpose, it is best to describe the circuit with *state equations*, to be discussed in the next chapter.

Problems

2.1. Prove Theorem 2.2, the superposition theorem.

2.2. Prove Theorem 2.4 on maximally independent *KCL* equations.

2.3. Show that in a system of two coupled inductors, the coupling coefficient is not larger than one in absolute value. See Theorem 2.7.

2.4. How would you extend Tellegen's theorem to circuits containing multi-terminal elements such as bipolar transistors and MOSFET? Consider an m-terminal element. Select one of the terminals as the reference terminal. Define $m-1$ terminal voltages, one from each of the non-reference terminals with respect to the reference terminal. Assign a terminal current to each of the $m-1$ non-reference terminals. Let $\mathbf{v}$ and $\mathbf{i}$ be the row vectors of terminal voltages and currents, respectively, of the circuit. (a) Show that $\mathbf{v}\,\mathbf{i}^T = 0$. (b) Let $\mathbf{v_a}$ and $\mathbf{v_b}$ be the vectors of terminal voltages of two topologically identical circuits A and B, respectively. Let $\mathbf{i_a}$ and $\mathbf{i_b}$ be the corresponding vectors of terminal currents. Show that $\mathbf{v_a}\,\mathbf{i_b}^T = 0$ and $\mathbf{v_b}\,\mathbf{i_a}^T = 0$.

2.5. Figure 2.4 shows a low-pass filter which can be used to remove the high frequency components of a signal. Let the elements be $R = 1000\,\Omega$, $C = 0.01592\,\mu\mathrm{F}$, and $L = 0.03183\,\mathrm{H}$. Suppose the input is $E(t) = \sin 2\pi f_1 t + 0.1 \sin 2\pi f_2 t$, with $f_1 = 5 \times 10^3$ Hz and $f_2 = 5 \times 10^4$ Hz. (a) Compute and plot the output voltage $v_3(t)$ over about four periods of the low frequency component of the input. Use Backward Euler method to solve the modified node equations. The output is shown in Fig. 2.5. (b) Lower the high frequency f_2 and see how low it can be before filtering action becomes ineffective. (c) Verify Tellegen's theorem at two different time points.

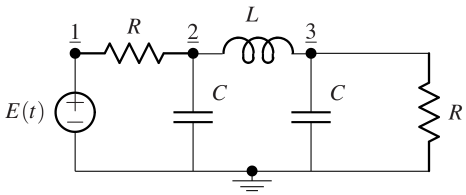

Fig. 2.4 A low pass filter

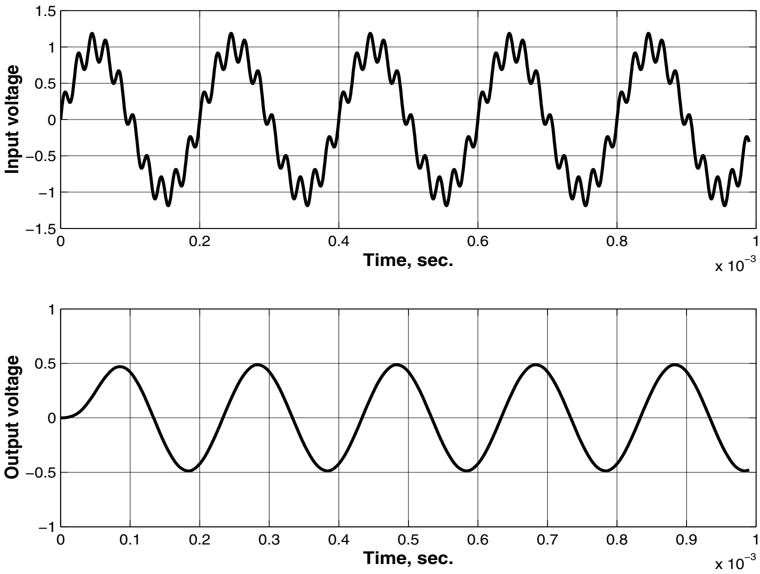

Fig. 2.5 Filtering of high frequency component of a signal by a low pass filter

2.6. A low-pass filter can also be used to filter out high frequency noise. Consider the circuit of the last problem. Let the input be $E(t) = \sin 2\pi f_1 t + n(t)$, where $n(t)$ is the noise. At each discrete time point, its amplitude is a random number whose value is normally distributed with mean 0 and variance 0.1.[3] Compute and plot the output voltage. A typical output is shown in Fig. 2.6. Repeat the problem with random numbers uniformly distributed over $[-0.1, 0.1]$.[4]

2.7. Inductors are bulky and expensive. In practice, we often replace an inductor with a resistor if the performance is acceptable. Repeat Problem 2.5 and Problem

[3] In MATLAB®, the command $y = m + \sigma * randn$ generates a random number taken from a universe of numbers which are normally distributed with mean m and variance σ^2 .

[4] In MATLAB®, the command $y = a + (b - a) * rand$ generates a random number taken from a universe of numbers which are uniformly distributed over $[a, b]$.

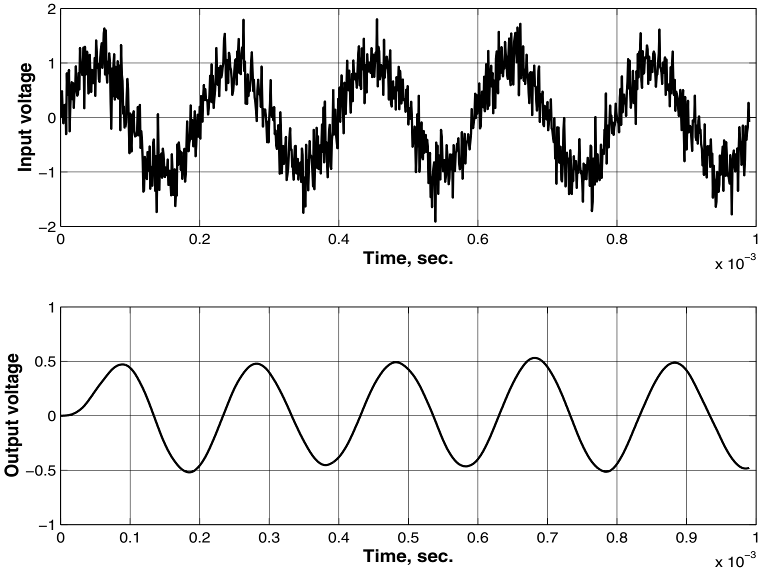

Fig. 2.6 Filtering of Gaussian noise by a third order low-pass filter

2.6, replacing the inductor with a resistor of $2\,\mathrm{k}\Omega$. What can you say about the performance in each case?

2.8. Figure 2.7 shows a high-pass filter. Let the element values be $R = 10^3\ \Omega$, $C = 0.01592\,\mu\mathrm{F}$, and $L = 0.007958\,\mathrm{H}$. Let the input voltage be $E(t) = 0.3\sin 2\pi f_1 t + \sin 2\pi f_2 t$, with $f_1 = 5\times 10^3$ Hz and $f_2 = 5\times 10^4$ Hz. (a) Compute and plot the output voltage $v_4(t)$ over about two periods of the low frequency component of the input. Use Backward Euler method to solve the modified node equations. The output is shown in Fig. 2.8. (b) Increase f_1 and see how high it can be before filtering action becomes ineffective. (c) Verify Tellegen's theorem at two different time points.

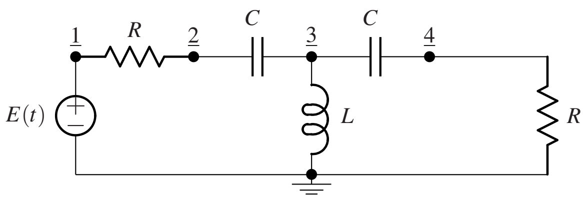

Fig. 2.7 A high-pass filter to remove a low frequency component of a signal

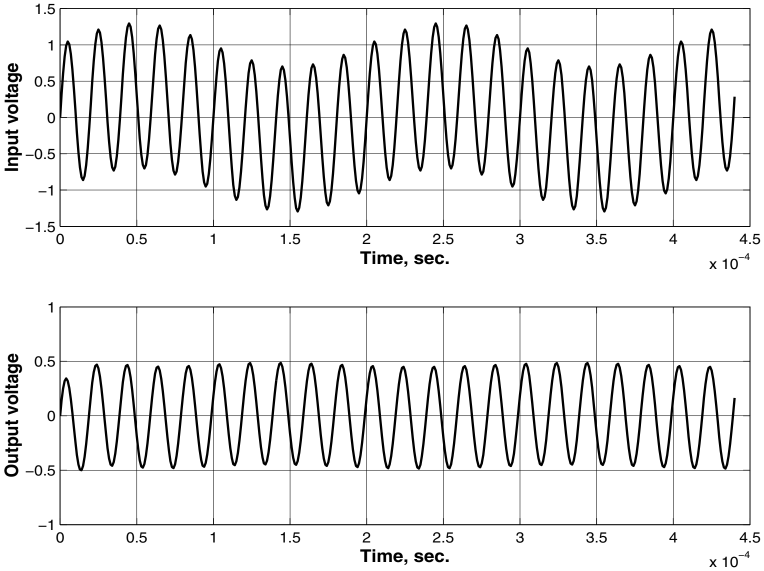

Fig. 2.8 Filtering of low frequency component of a signal by a high pass filter

2.9. Apply a signal with Gaussian noise to a high-pass filter. What do you expect to see at the output? In the high-pass filter of the last problem, let $E(t) = \sin 2\pi f_2 t + n(t)$, where the noise signal $n(t)$ is as specified in Problem 2.6. Compute and plot the output, which should be as shown in Fig. 2.9.

2.10. Repeat Problem 2.8, except that we replace the inductor with a resistor of $500\,\Omega$. What can you say about the performance of the circuit compared to that of the original circuit?

2.11. Figure 2.10 is a third order band-pass filter which we will study in future chapters. Let the input voltage $E(t)$ be:

$$E(t) = \sin t + \sin 4t + \sin 8t \ . \tag{2.68}$$

It consists of three sinusoidal components of equal amplitude but different frequencies at 1, 4, and 8 rad/s. Compute the output voltage $v_{out}(t)$ over sufficiently long period of time to show the filtering function of the circuit.

2.12. The circuit of Fig. 2.11 with switch S closed has been at rest for a long time. At $t = 0$, S opens. Find the largest value of C such that the maximum output voltage is not less than 50 V. $L_1 = 100$ mH, $L_2 = 400$ mH, $M = -160$ mH, $R = 1\,\mathrm{K}\Omega$, $G = 1 \times 10^{-6}$ S, and $E = 1$ V.

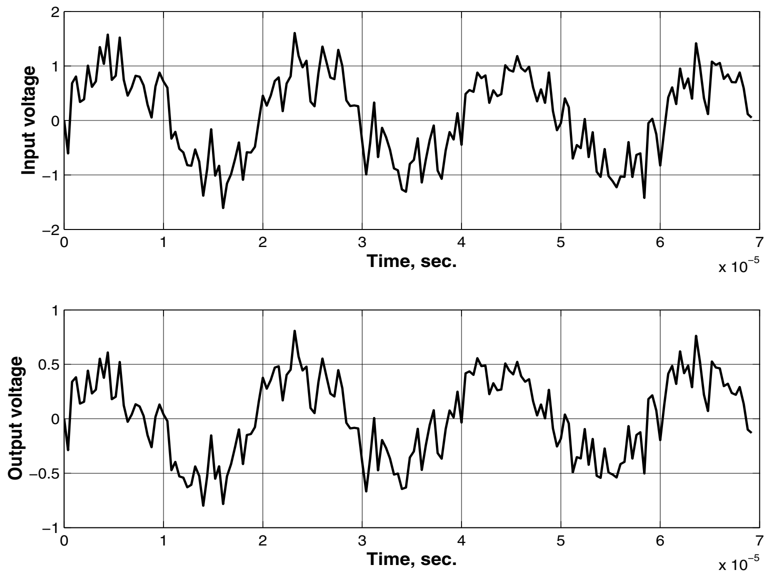

Fig. 2.9 A noisy high frequency signal passes through a high pass filter essentially unaltered

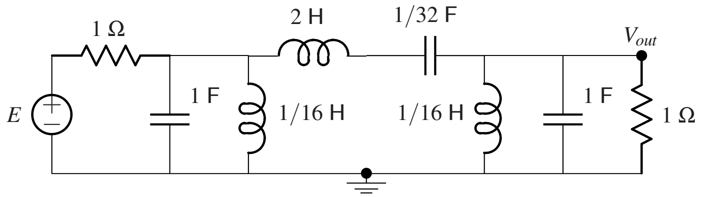

Fig. 2.10 A third order band-pass filter

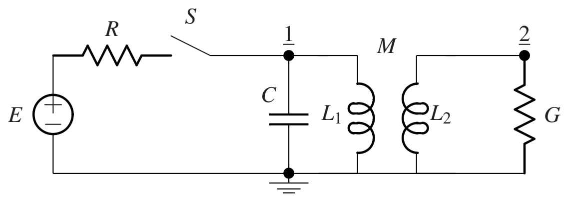

Fig. 2.11 A circuit to generate a high voltage pulse

2.13. An interconnect in an integrated circuit is often modeled as an *RC* ladder circuit as shown in Fig. 2.12. Assume five sections are sufficient to represent a line, with $R = 1\ \Omega$ and $C = 0.1$ F. Use the backward Euler method to compute the output voltage for an input pulse of height 1 V and duration of 1 second. The result should be as shown in Fig. 2.13. Notice the dispersion (delay and widening) of the input pulse and reduction of amplitude at the output. Repeat the problem for a 10-section ladder. Vary the *RC* product to see how it affects the response.

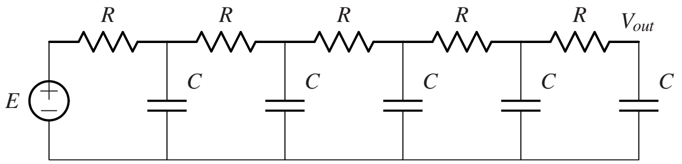

Fig. 2.12 A five-section *RC* ladder circuit to represent a length of interconnect in an integrated circuit

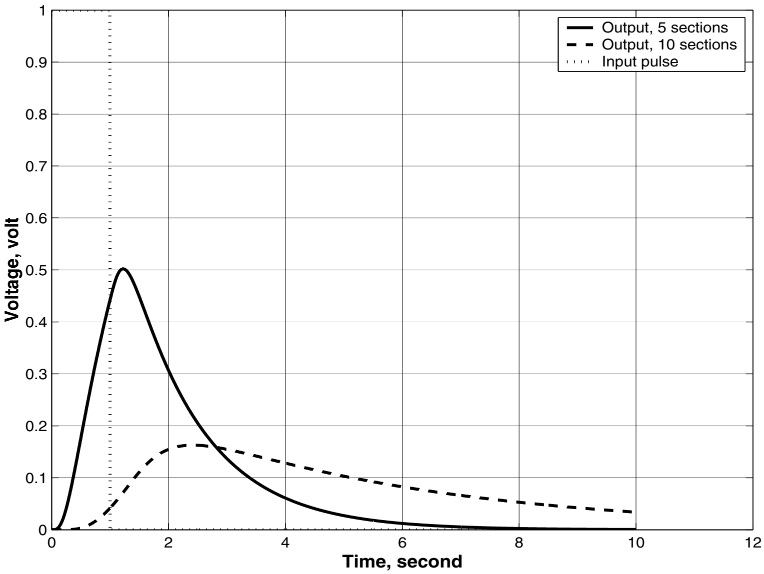

Fig. 2.13 Response of a five-section and a ten-section *RC* ladder to an input pulse. Note the dispersion of the input and reduction of amplitude at the output

Chapter 3
Circuit Dynamics

A circuit is a dynamical system. Circuit variables change with time in response to an applied input, or in the absence of any input, they evolve from their initial values into some functions of time. We speak of the *state* of a circuit at a given time as the set of voltages and currents of the circuit at that time. At a later time, the voltages and currents take on different values and the state changes. As we saw in the last chapter, not all circuit variables are necessary in order to describe the behavior of a circuit. The minimal set of variables needed to describe the circuit dynamics are called the *state variables*, and they satisfy a set of differential equations known as *state equations*. The solution of the state equations prescribes how the state changes over time, and the time history is best represented as a set of points, or a *trajectory*, in a space spanned by the state variables, called the *state space*.

In this chapter we study the dynamics of linear circuits in the state space. We show how state equations are formulated and solved, and we deduce the kind of time functions that the state variables may assume in response to initial conditions. The response to an arbitrary input is formally expressed in closed form as a convolution integral.

3.1 State equations

State equations for circuits were first introduced by Bashkow in 1957 [3]. He noted that in a circuit, it is the rate of change of the capacitor voltages and inductor currents that gives rise to its dynamics. It follows that we should use these circuit variables as state variables, provided they are linearly independent.

3.1.1 A simple example

To fix ideas, consider a series *RLC* circuit as shown in Fig. 3.1. Writing circuit

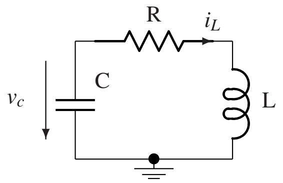

Fig. 3.1 A simple series *RLC* circuit to illustrate trajectory in the state space

equations in terms of the capacitor voltage v_c and inductor current i_L, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_c \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix}. \tag{3.1}$$

To complete the description, we need to specify the initial conditions $v_c(0)$ and $i_L(0)$. We will take up formal solution of state equations shortly. For now, we may use any method, for example, the method of undetermined coefficients, to obtain the solution, which is:

$$v_c(t) = v_c(0)\mathrm{e}^{-\alpha t}\cos\beta t + \left(\frac{-1}{C\beta} i_L(0) + \frac{\alpha}{\beta} v_c(0)\right)\mathrm{e}^{-\alpha t}\sin\beta t\,, \tag{3.2}$$

$$i_L(t) = i_L(0)\mathrm{e}^{-\alpha t}\cos\beta t + \left(\frac{-\alpha}{\beta} i_L(0) + \frac{1}{L\beta} v_c(0)\right)\mathrm{e}^{-\alpha t}\sin\beta t\,, \tag{3.3}$$

where $\alpha = R/2L$, and $\beta = \sqrt{1/LC - \alpha^2}$. This is the response of the circuit to initial conditions. To display the dynamics of the circuit, we start from the initial point $(v_c(0), i_L(0))$ and plot the locus of points of $(v_c(t), i_L(t))$ for $t \geq 0$ in the $v_c - i_L$ plane. Fig. 3.2 shows the trajectories from two different initial conditions. Each trajectory is a decaying spiral that approaches the origin as $t \to \infty$.

3.1.2 Uniqueness of solution

Equation (3.1) is a linear ordinary differential equation of the form:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\,\mathbf{x} + e(t)\,, \tag{3.4}$$

where $\mathbf{x}$ is the vector of state variables, A is a constant matrix, and $e(t)$ is a vector of excitations. It is a special case of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t)\,, \qquad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0\,. \tag{3.5}$$

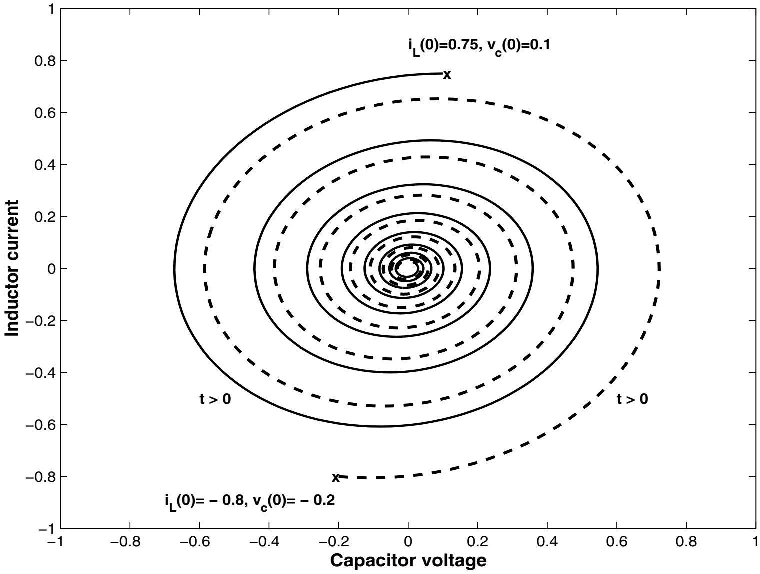

Fig. 3.2 Trajectories of state from two initial conditions in circuit of Fig. 3.1

Our problem is to find $\mathbf{x}(t)$ for $t > 0$ that satisfies the differential equation and initial conditions, and it is known as an *initial value problem*.

A great deal is known about initial value problems. For our purpose, the most important property is that in a region where $\mathbf{f}(\mathbf{x},t)$ is continuous and in which Lipschitz's condition[1] holds, a solution of Eq. (3.5) exists and it is unique. For circuits, there is always a unique solution.

An immediate consequence of the uniqueness of solution is that the trajectories originated from two different initial conditions do not intersect.

3.1.3 Normal form

Suppose we add a current source $J(t)$ across the capacitor in the circuit of Fig. 3.1. The state equations become

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_c \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}\begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 1/C & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} J(t) \\ 0 \end{bmatrix}, \tag{3.6}$$

which is of the form

[1] Informally, Lispschitz's condition holds in a region D if and only if there exists a constant L such that for all $(\mathbf{x},t)$ and $(\mathbf{x}',t)$ in D, $|\mathbf{f}(\mathbf{x},t) - \mathbf{f}(\mathbf{x}',t)| \leq L|\mathbf{x}-\mathbf{x}'|$.

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x} + B\mathbf{u}(t)\,, \tag{3.7}$$

where B is a constant matrix and $\mathbf{u}(t)$ is a vector of sources. Equation (3.7) is in what is known as the *normal form* of the state equations. Once the state variables are known, all the circuit variables can be expressed as linear combinations of them. For example, the responses: capacitor current $i_c(t)$, inductor voltage $v_L(t)$, and resistor voltage $v_R(t)$, can be expressed as

$$\begin{bmatrix} i_c \\ v_L \\ v_R \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -R \\ 0 & R \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J(t) \\ 0 \end{bmatrix}. \tag{3.8}$$

Formally, if $\mathbf{y}(t)$ is some vector of circuit responses or output, we can write

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)\,, \tag{3.9}$$

where C and D are constant matrices. Matrices A, B, C, and D are sometimes called the A, B, C, D matrices of the dynamic system which is the circuit.

3.2 Independent state variables

The dynamics of a circuit is determined by the rate of change of capacitor voltages and inductor currents. It appears that there are as many state equations as there are capacitors and inductors. However, it is possible that these elements are so connected that a linear relation exists among the capacitor voltages or among the inductor currents. If so, the number of independent state equations is reduced. In this section, we examine these issues.

3.2.1 Circuit with a capacitor loop

Consider the circuit of Fig. 3.3. There are three capacitor voltages that form a closed loop. By *KVL*, there is a linear relation among them:

$$-v_1 + v_3 + v_2 = 0\,. \tag{3.10}$$

So only two of the three are independent state variables. Choosing v_1 and v_2 as independent variables and writing *KCL* equations at nodes 1 and 2, we find the state equations to be

$$\begin{bmatrix} C_1 + C_3 & -C_3 \\ -C_3 & C_2 + C_3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -G_1 & 0 \\ 0 & -G_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} G_1 E(t) \\ 0 \end{bmatrix}. \tag{3.11}$$

The matrix on the left is a capacitance matrix C of the circuit when all the resistors are replaced by open circuits. Its inverse C^{-1} exists. In normal form, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C^{-1}\begin{bmatrix} -G_1 & 0 \\ 0 & -G_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + C^{-1}\begin{bmatrix} G_1 E(t) \\ 0 \end{bmatrix}. \tag{3.12}$$

The circuit is a second order system even though there are three capacitors. Only two of the three initial values of the capacitor voltages can be arbitrarily specified, since *KVL* holds for all times.

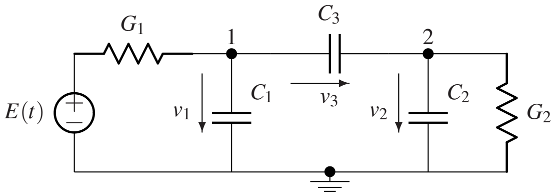

Fig. 3.3 A second order *RC* circuit with a loop of three capacitors

3.2.2 Circuit with an inductor cutset

Now consider the circuit of Fig. 3.4. There are three inductors that form a cut set,[2] so that there exists a linear relation among the inductor currents, namely,

$$i_1 - i_2 - i_3 = 0 \tag{3.13}$$

and only two initial values can be arbitrarily specified. Choosing i_1 and i_2 as independent state variables and writing *KVL* equations for the fundamental loops associated with the tree $\{R_1, E(t), L_3, R_2\}$,[3], [4] we get

$$\begin{bmatrix} L_1+L_3 & -L_3 \\ -L_3 & L_2+L_3 \end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -R_1 & 0 \\ 0 & -R_2 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} E(t) \\ 0 \end{bmatrix}. \tag{3.14}$$

The state equations take the normal form after the system is multiplied by the inverse of the inductance matrix. The inductance matrix is obtained when we write *KVL* equations for the fundamental loops of the circuit with all resistors replaced by short circuits.

---

[2] A cut set is a set of elements whose removal leaves the circuit with two disconnected parts and no proper subset of the set has this property.

[3] A tree is a set of elements containing all the nodes of a circuit but no loops.

[4] A fundamental loop associated with a tree is a set of elements forming a loop and containing some elements of the tree and exactly one non-tree element, called a chord of the tree.

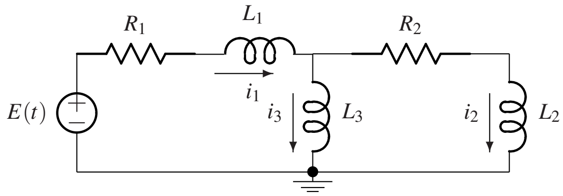

Fig. 3.4 A second order *RL* circuit with a cutset of three inductors

3.3 Order of state equations

The examples of the last section suggest that the number of independent state variables depends on the topology of the circuit. We define the *order* of a circuit as the minimum number of independent state variables, or the minimum number of independent state equations, necessary to describe the dynamics completely. It is the total number of capacitors and inductors less the total number of linear relations among the capacitor voltages and inductor currents.

3.3.1 Capacitor loops and inductor cut sets

Given a circuit N, let us define a *C-subcircuit* as a subcircuit of N obtained by removing all but the capacitors and voltage sources. The C-subcircuit may consist of separate connected parts. Define a *C-fundamental loop* as a fundamental loop of a connected part of the C-subcircuit. In each C-fundamental loop, exactly one capacitor voltage can be expressed as a linear combination of the rest of the capacitor voltages in the loop. For each C-fundamental loop, the number of independent state variables is reduced by one.

Similarly, we define an *L-subcircuit* as a subcircuit of N obtained by shorting all but the inductors and current sources. Remove any shorted inductors and shorted current sources. Define an *L-fundamental cut set* as a fundamental cut set[5] of the remaining L-subcircuit. In each L-fundamental cut set, exactly one inductor current can be expressed as a linear combination of the rest of the inductor currents in the cut set, and the number of independent state variables is reduced by one.

3.4 Formulation of state equations

State equations are expressions of capacitor currents and inductor voltages in terms of capacitor voltages and inductor currents. The general principle of formulation of

[5] A fundamental cut set with respect to a tree is a set of elements consisting of chords of the tree and exactly one tree element.

state equations is to replace each capacitor with a voltage source and each inductor with a current source and then solve for the capacitor currents and inductor voltages to obtain the state equations, taking into account any linear relations among the variables.

3.4.1 Circuits without capacitor loops or inductor cut sets

Consider the circuit of Fig. 3.5(a). After replacing the capacitors and inductors with their respective sources, we get a resistor circuit shown in (b). Its node equations are

$$\begin{bmatrix} g_1+g_2 & -g_2 & 0 & 0 & 1 & 0 & 0 \\ -g_2 & g_2+g_5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & g_3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & g_4 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{1n} \\ v_{2n} \\ v_{3n} \\ v_{4n} \\ i_{c1} \\ i_{c2} \\ v_\ell \end{bmatrix} = \begin{bmatrix} J \\ 0 \\ -i_\ell \\ i_\ell \\ v_{c1} \\ v_{c2} \\ 0 \end{bmatrix}. \tag{3.15}$$

To simplify algebra, let $g_1 = g_2 = g_3 = g_4 = g_5 = 1\mathsf{S}$, $L = 1\mathsf{H}$, and $C_1 = C_2 = 1\mathsf{F}$. Solving for the inductor voltage and capacitor currents, we obtain:[6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_{c1} \\ v_{c2} \\ i_\ell \end{bmatrix} = \begin{bmatrix} -5/8 & 1/8 & 3/4 \\ 1/8 & -5/8 & -3/4 \\ -3/4 & 3/4 & -1/2 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ i_\ell \end{bmatrix} + \begin{bmatrix} 3/8J \\ 1/8J \\ 1/4J \end{bmatrix}. \tag{3.16}$$

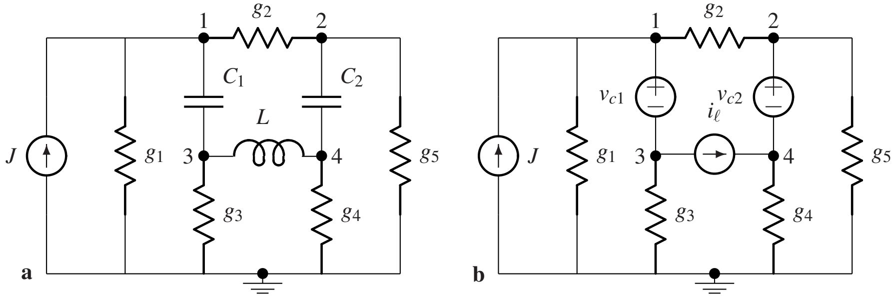

Fig. 3.5 **a** A circuit without capacitor loops or inductor cutsets. **b** Resistor circuit after replacement of capacitors by voltage sources and inductor by current source

[6] In MATLAB®, $Ax = b$, with elements of b declared to be symbolic, is solved by invoking $x = A\backslash b$.

3.4.2 Circuits with capacitor loops

Consider the circuit of Fig. 3.6. There are two C-fundamental loops. Choose v_{c1}, v_{c2}, and v_{c3} as state variables. The chord capacitor currents are expressible as

$$i_{c4} = c_4 \frac{\mathrm{d}v_{c1}}{\mathrm{d}t} - c_4 \frac{\mathrm{d}v_{c2}}{\mathrm{d}t}, \tag{3.17}$$

$$i_{c5} = c_5 \frac{\mathrm{d}v_{c2}}{\mathrm{d}t} - c_5 \frac{\mathrm{d}v_{c3}}{\mathrm{d}t}. \tag{3.18}$$

Writing *KCL* at nodes 2, 3 and 4, and making substitutions, we find:

$$\begin{bmatrix} c_1+c_4 & -c_4 & 0 \\ -c_4 & c_2+c_4+c_5 & -c_5 \\ 0 & -c_5 & c_3+c_5 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v_{c1} \\ v_{c2} \\ v_{c3} \end{bmatrix} = \begin{bmatrix} -g_1-g_4 & 0 & 0 \\ 0 & -g_2 & 0 \\ 0 & 0 & -g_3 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ v_{c3} \end{bmatrix} + \begin{bmatrix} g_4 E(t) \\ 0 \\ 0 \end{bmatrix}. \tag{3.19}$$

The matrix on the left is the capacitance matrix of the circuit when all the resistors are open-circuited.

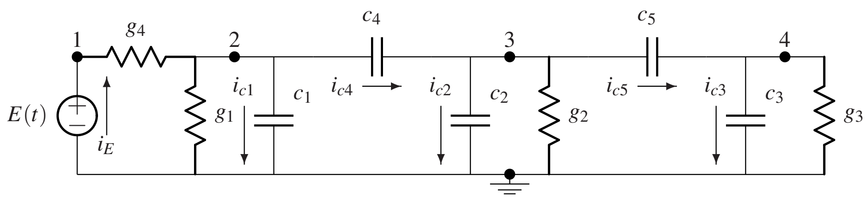

Fig. 3.6 Formulation of state equations for a circuit with two fundamental capacitor loops

3.4.3 Circuits with inductor cut sets

The circuit of Fig. 3.7 contains two inductor cut sets. Choose a tree to include R, C, L_4 and L_5. The two L-fundamental cut sets are $\{L_1, L_2, L_4\}$ and $\{L_3, L_2, L_5\}$, so that $i_4 = i_1 - i_2$ and $i_5 = i_3 + i_2$.

In constructing the state equations, L_4 and L_5 are replaced with their respective voltage sources of value:

$$v_4 = L_4 \frac{\mathrm{d}i_1}{\mathrm{d}t} - L_4 \frac{\mathrm{d}i_2}{\mathrm{d}t}, \qquad v_5 = L_5 \frac{\mathrm{d}i_3}{\mathrm{d}t} + L_5 \frac{\mathrm{d}i_2}{\mathrm{d}t}. \tag{3.20}$$

After replacing the chord inductors with current sources and the capacitor with a voltage source, we find the state equations to be

$$\begin{bmatrix} L_1+L_4 & -L_4 & 0 & 0 \\ -L_4 & L_2+L_4+L_5 & L_5 & 0 \\ 0 & L_5 & L_3+L_5 & 0 \\ 0 & 0 & 0 & C \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ v_c \end{bmatrix} = \begin{bmatrix} -R & 0 & R & -1 \\ 0 & 0 & 0 & 1 \\ R & 0 & -R & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ v_c \end{bmatrix} + \begin{bmatrix} RJ \\ 0 \\ RJ \\ 0 \end{bmatrix} . \tag{3.21}$$

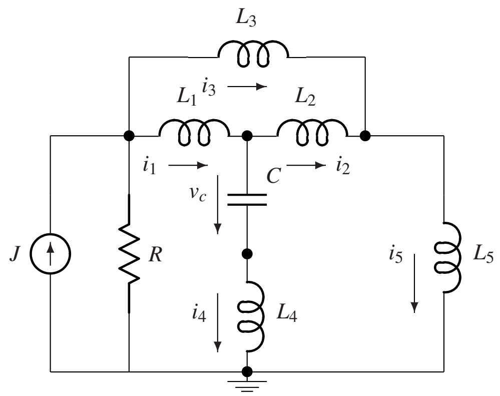

Fig. 3.7 Formulation of state equations for a circuit with two fundamental inductor cut sets

3.4.4 Remarks

The general procedure is clear. We first select a tree to include as many capacitors as tree elements and as many inductors as chords as possible. Replace each capacitor with a voltage source and each inductor with a current source. Solve for the capacitor currents and inductor voltages from the resultant resistor circuit. After substituting the linear relations, if any, among the variables, we obtain the state equations. This procedure can be formalized as reported in Brown [11], Bryant [13], and Branin [10].

3.5 Solution of state equations

Our starting point is Eq. (3.7), the normal form of state equations:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + B\mathbf{u}(t)\,, \tag{3.22}$$

with a vector of known initial conditions $\mathbf{x}(0)$. Let $H(t)$ be a matrix defined as

$$H(t) = e^{At}\,, \tag{3.23}$$

where formally,

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots\,. \tag{3.24}$$

$H(t)$ has a few important properties:

1. $H(0) = I$.
2. $H(t)$ satisfies the homogeneous equation:

$$\frac{dH(t)}{dt} = A\,H(t)\,. \tag{3.25}$$

3. $H(-t) = H^{-1}(t)$.

The solution to Eq. (3.22) is then

$$\mathbf{x}(t) = H(t)\mathbf{x}(0) + \int_0^t H(t)\,H(\tau)^{-1}B\,\mathbf{u}(\tau)\,d\tau\,, \tag{3.26}$$

as can be verified by direct substitution into Eq. (3.22). It remains to find $H(t)$.

Assume the eigenvalues of A are all distinct. Then $H(t)$ is given by

$$H(t) = T\,e^{\Lambda t}\,T^{-1}\,, \tag{3.27}$$

where T is a matrix of eigenvectors and Λ is a diagonal matrix of eigenvalues. Let the eigenvalues be $\lambda_1, \lambda_2, \cdots, \lambda_n$. They are the roots of the characteristic equation:

$$det(A - \lambda I) = 0\,. \tag{3.28}$$

Then the columns of T, T_k, satisfy

$$(A - \lambda_k I)T_k = 0 \tag{3.29}$$

for each $k = 1, \cdots, n$.[7] The exponential term is

[7] In MATLAB®, $[V,D] = eig(A)$ places the eigenvalues of A in a diagonal matrix D and their corresponding eigenvectors in V.

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}. \tag{3.30}$$

3.5.1 Impulse response

Because of Property (3), the solution can also be written as

$$\mathbf{x}(t) = H(t)\mathbf{x}(0) + \int_0^t H(t-\tau)B\mathbf{u}(\tau)\,d\tau \tag{3.31}$$

and it consists of two parts. One is due to initial conditions $\mathbf{x}(0)$ alone ($\mathbf{u}(t) \equiv 0$) and is called the *homogeneous solution*.The other is the convolution of a linear combination of the excitation $\mathbf{u}(t)$ with matrix $H(t)$ and is called the *particular integral*.

Assume $\mathbf{u}(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T$. Consider the special case where all the initial conditions are zero and all but one of the components, $u_k(t)$, are zero. Without loss of generality, we assume the coefficient of $u_k(t)$ is unity. Suppose $u_k(t)$ is a unit impulse $\delta(t)$. Substitution into Eq. (3.31) yields

$$\mathbf{x}(t) = \int_0^t H(t-\tau)\,[0,\cdots,\delta(\tau),0,\cdots,0]^T\,d\tau = H_k(t), \tag{3.32}$$

where $H_k(t)$ is the kth column of $H(t)$, and we obtain the following theorem.

Theorem 3.1. *The matrix $H(t)$ is a matrix of impulse responses, with the* k*th column being the response of the circuit to a unit impulse applied as the* k*th component of the input when all the initial conditions and all other components of the input vector are zero.*

We shall call $H(t)$ the *impulse response matrix*. On the other hand, suppose the input $\mathbf{u}(t) \equiv 0$. Then the solution $\mathbf{x}(t) = H(t)\mathbf{x}(0)$, and we can interpret the kth column of $H(t)$ as the response due to an initial condition of value one of the kth state variable. So we have

Theorem 3.2. *The matrix $H(t)$ is a matrix of responses due to initial conditions of the state variables, with the* k*th column being the response to a unit initial condition of the* k*th state variable, all other initial conditions being zero.*

Next let us look at the eigenvalues. They are roots of the characteristic polynomial $det[A-\lambda I]$. Since A is a matrix of real constants, the coefficients of the polynomial in λ are real and we have

Theorem 3.3. *The eigenvalues of the state equations of a circuit are real or occur in complex conjugate pairs.*

By Eq. (3.27) and (3.30), we see that each term of $H(t)$ is a linear combination of exponential terms. Since circuit responses cannot grow without limit in time, the real part of each eigenvalue must be non-positive. It follows that real eigenvalues give rise to decaying exponential terms and complex eigenvalues give rise to damped or undamped sinusoidal terms. As a result, we have

Theorem 3.4. *The impulse response of a circuit with distinct eigenvalues is of the form*

$$h(t) = \sum_k (a_k e^{\alpha_k t} \sin \beta_k t + b_k e^{\alpha_k t} \cos \beta_k t), \tag{3.33}$$

where $\alpha_k \leq 0$.

3.5.2 Examples

Example 3.1. Consider the circuit of Fig. 3.5 whose state equations are given in Eq. (3.16). Let us find the response due to an initial condition $v_{c1}(0) = 1\mathsf{V}$, all other initial conditions being zero. Plot the trajectory of the state in the state space.

Solution 3.1. We find the eigenvalues to be

$$\lambda_1 = -0.500, \qquad \lambda_2 = -0.625 + 1.0533j, \qquad \lambda_3 = -0.625 - 1.0533j, \tag{3.34}$$

and the eigenvector matrix is

$$T = \begin{bmatrix} 0.7071 & -0.0589 + 0.4965j & -0.0589 - 0.4965j \\ 0.7071 & 0.0589 - 0.4965j & 0.0589 + 0.4965j \\ 0 & -0.7071 & -0.7071 \end{bmatrix}. \tag{3.35}$$

The first column of $H(t)$ is what we want and the state variables are found to be:

$$v_{c1}(t) = 0.5e^{-0.5t} + 0.5e^{-0.625t} \cos 1.0533t - 0.0594e^{-0.625t} \sin 1.0533t, \tag{3.36}$$

$$v_{c2}(t) = 0.5e^{-0.5t} - 0.5e^{-0.625t} \cos 1.0533t + 0.0594e^{-0.625t} \sin 1.0533t, \tag{3.37}$$

$$i_\ell(t) = -0.712e^{-0.625t} \sin 1.0533t. \tag{3.38}$$

Each is a combination of decaying exponential and damped sinusoids. Fig. 3.8 shows the trajectory of the state in the state space. Starting at the initial point $\{v_{c1}(0), v_{c2}(0), i_\ell(0)\} = \{1, 0, 0\}$, the trajectory "spirals" to zero.

Example 3.2. In the circuit of Fig. 3.6, let $g_k = k$ S, and $c_k = k$ F, $k = 1, \ldots, 5$. The state equations in normal form becomes:

$$\frac{d}{dt} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1.6845 & -0.3422 & -0.3209 \\ -0.8556 & -0.4278 & -0.4011 \\ -0.5348 & -0.2674 & -0.6257 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 1.3476E(t) \\ 0.6844E(t) \\ 0.4280E(t) \end{bmatrix}. \tag{3.39}$$

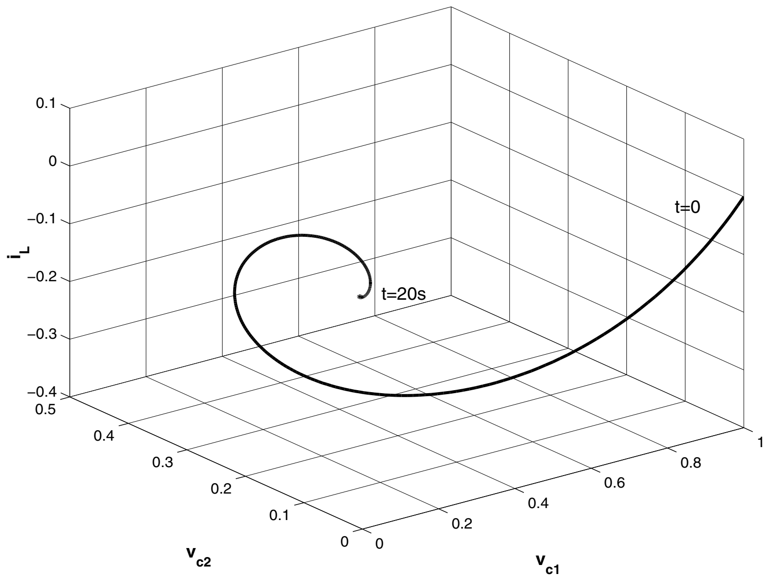

Fig. 3.8 Trajectory of state in state space of an *RLC* circuit

Let us find the output voltage $v_{4n} = v_3$ for some input $E(t)$, assuming all initial conditions are zero.

Solution 3.2. The eigenvalues are $\lambda_1 = -2.0627$, $\lambda_2 = -0.5279$, and $\lambda_3 = -0.1473$, all negative real. This is not an accident. It will be shown in a later chapter that the eigenvalues of an RC circuit are always negative real. The solution sought is the convolution of the last row of $H(t)$ with the input vector in Eq. (3.39) and we get:

$$v_{4n}(t) = \int_0^t [0.6192\mathrm{e}^{\lambda_1(t-\tau)} - 0.2040\mathrm{e}^{\lambda_2(t-\tau)} + 0.0128\mathrm{e}^{\lambda_3(t-\tau)}]E(\tau)\,\mathrm{d}\tau\,. \quad (3.40)$$

If the input $E(t)$ is expressible in terms of elementary functions, it is possible to obtain a closed form expression of the convolution integral. For an arbitrary $E(t)$, a closed form solution may not exist and we must resort to numerical evaluation.

Example 3.3. Find the response of the circuit below to an initial voltage $v_{c1}(0) = 1$V, all other initial conditions being zero.

Solution 3.3. The state equations are:

$$\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_{c1} \\ v_{c2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_{c1} \\ v_{c2} \end{bmatrix}. \quad (3.41)$$

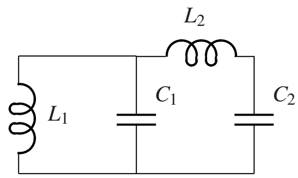

Fig. 3.9 A fourth order lossless circuit. All of its eigenvalues are imaginary

Let $L_1 = 1\text{H}$, $L_2 = 12/5\text{H}$, $C_1 = 1/6\text{F}$, and $C_2 = 5/18\text{F}$. The eigenvalues are $\pm j$ and $\pm 3j$. The response is the third column of the matrix $H(t)$:[8]

$$i_{L1}(t) = -\frac{5}{16}\sin 3t - \frac{1}{16}\sin t\,, \tag{3.42}$$

$$i_{L2}(t) = \frac{5}{32}\sin 3t - \frac{5}{96}\sin t\,, \tag{3.43}$$

$$v_{c1}(t) = \frac{15}{16}\cos 3t + \frac{1}{16}\cos t\,, \tag{3.44}$$

$$v_{c2}(t) = -\frac{3}{16}\cos 3t + \frac{3}{16}\cos t\,. \tag{3.45}$$

The voltages and currents are all sums of undamped sinusoids. In a later chapter, we will show that in a lossless circuit, the eigenvalues are all imaginary, and $H(t)$ consists of sine and cosine terms. The state space is four-dimensional. Fig. 3.10 shows the projection of the state trajectory in the $i_{L1} - v_{c1} - v_{c2}$ space. It is a closed curve with no beginning or end, meaning it is periodic.

3.6 Repeated eigenvalues

It is possible that a circuit has one or more eigenvalues of multiplicity greater than one. If we follow the same procedure to obtain the solution as the case of distinct eigenvalues, we will find that the eigenvalue matrix becomes singular and the impulse response matrix $H(t)$ cannot be expressed as in Eq. (3.27).

Consider the following homogeneous state equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{3.46}$$

The eigenvalue is -2 and is repeated. We note the first equation is decoupled from the second, and its solution is

[8] In MATLAB®, $H = expm(At)$, where A is a matrix of simple rational numbers and t declared symbolic, will obtain $H(t)$ in analytic form.

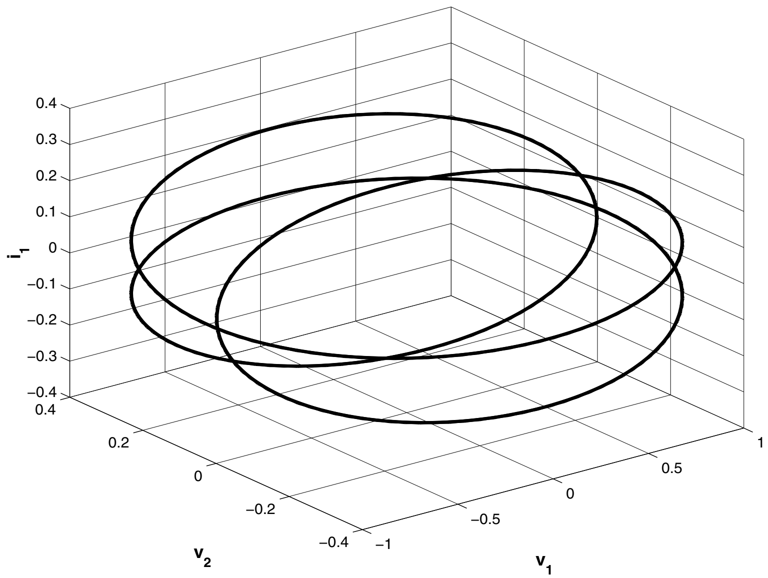

Fig. 3.10 A periodic trajectory in a 3-D state space of a fourth order circuit

$$x_1(t) = x_1(0)e^{-2t}. \tag{3.47}$$

Substituting into the second equation and regarding $x_1(t)$ as an excitation, we get

$$\begin{aligned} x_2(t) &= x_2(0)e^{-2t} + \int_0^t e^{-2(t-\tau)} x_1(0)e^{-2\tau}\, d\tau \\ &= x_2(0)e^{-2t} + x_1(0)te^{-2t}. \end{aligned} \tag{3.48}$$

Thus the homogeneous solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} & 0 \\ te^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \tag{3.49}$$

and the impulse response matrix is

$$H(t) = \begin{bmatrix} e^{-2t} & 0 \\ te^{-2t} & e^{-2t} \end{bmatrix}. \tag{3.50}$$

In general, if the state equations are in the "elementary Jordan" form:

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda_0 \end{bmatrix} x = J_0 x, \tag{3.51}$$

then the impulse response matrix is given by

$$H(t) = \begin{bmatrix} e^{\lambda_0 t} & 0 & 0 & \cdots & 0 \\ te^{-\lambda_0 t} & e^{\lambda_0 t} & 0 & \cdots & 0 \\ \frac{t^2}{2!}e^{\lambda_0 t} & te^{-\lambda_0 t} & e^{\lambda_0 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{t^{m-1}}{(m-1)!}e^{\lambda_0 t} & \cdots & & & e^{\lambda_0 t} \; . \end{bmatrix} \tag{3.52}$$

The matrix J_0 is known as the elementary Jordan matrix.

Not every system matrix which has repeated eigenvalues is a Jordan matrix. However, by taking linear combinations of the state variables, the state equations can be transformed into this form. Consider the circuit of Fig. 3.11. Let us find the complete solution for $t \geq 0$ for an excitation given by: $E(t) = 4\text{V}$ for $t < 0$ and $E(t) = E'(t)$ for $t \geq 0$, where $E'(t)$ is some arbitrary function of time. Let

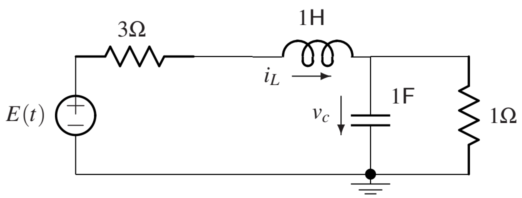

Fig. 3.11 A circuit with repeated eigenvalues: $\lambda_1 = \lambda_2 = -2$

$\mathbf{x} = [v_c, i_L]^T$. The initial conditions are: $x_1(0) = 1\text{A}$ and $x_2(0) = 1\text{V}$. The state equations are:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ E'(t) \end{bmatrix} = A\mathbf{x} + \mathbf{u}(t), \tag{3.53}$$

which have repeated eigenvalues equal to -2. The state matrix A is not in Jordan form. Let

$$\mathbf{y} = T\mathbf{x}, \tag{3.54}$$

where T is a transformation matrix. Substituting into Eq. (3.53), we have

$$\frac{d\mathbf{y}}{dt} = TAT^{-1}\mathbf{y} + T\mathbf{u}(t) \; . \tag{3.55}$$

We look for a transformation such that

$$TAT^{-1} = J_0 \,. \tag{3.56}$$

Expanding and equating coefficients, we find T to be

$$T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} . \tag{3.57}$$

Solving Eq. (3.55) with the initial conditions: $y_1(0) = 2$ and $y_2(0) = 3$, we get

$$y_1(t) = 2\mathrm{e}^{-2t} + \int_0^t \mathrm{e}^{-2(t-\tau)} E'(\tau)\,\mathrm{d}\tau \,, \tag{3.58}$$

$$y_2(t) = 3\mathrm{e}^{-2t} + 2t\,\mathrm{e}^{-2t} + \int_0^t [(t-\tau)+1]\,\mathrm{e}^{-2(t-\tau)} E'(\tau)\,\mathrm{d}\tau \,. \tag{3.59}$$

Transforming back to x, we have the complete solution given by

$$v_c(t) = x_1(t) = \mathrm{e}^{-2t} + 2t\,\mathrm{e}^{-2t} + \int_0^t (t-\tau)\mathrm{e}^{-2(t-\tau)} E'(\tau)\,\mathrm{d}\tau \,, \tag{3.60}$$

$$i_L(t) = x_2(t) = \mathrm{e}^{-2t} - 2t\,\mathrm{e}^{-2t} + \int_0^t [1-(t-\tau)]\,\mathrm{e}^{-2(t-\tau)} E'(\tau)\,\mathrm{d}\tau \,. \tag{3.61}$$

In the case where we have two repeated eigenvalues of multiplicities m_1 and m_2, we look for a transformation T such that

$$TAT^{-1} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} , \tag{3.62}$$

where J_1 and J_2 are Jordan matrices of order m_1 and m_2, respectively. Each subsystem is solved separately and the solutions are recombined by inverse transformation, to obtain the solution to the original system. We will omit all the details.

From the above, it is evident that

Observation 3.1 *The impulse response of a linear circuit is of the form:*

$$h(t) = \sum_k a_k t^{m_k-1} \mathrm{e}^{\alpha_k t} \sin \beta_k t + \sum_k b_k t^{m_k-1} \mathrm{e}^{\alpha_k t} \cos \beta_k t \,, \tag{3.63}$$

where m_k is the multiplicity of eigenvalue pair $\alpha_k \pm j\beta_k$.

3.7 Symbolic solution

That the solution of the state equations of a linear circuit can be obtained systematically in closed form, at least for excitations expressible in simple elementary functions, suggests that we should be able to express the solution symbolically. This is indeed the case. For example, the state equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x - 4y + 51 \sin t\,,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 3y\,, \tag{3.64}$$

with initial conditions $x(0) = 1$ and $y(0) = 0$ is solved by invoking the MATLAB® function $[x,y] = dsolve(..,..,..,..)$ to produce the solution

$$x = 1/2\,\mathrm{e}^{-3t}\cos 4t - 2\,\mathrm{e}^{-3t}\sin 4t + 1/2\,\cos t + 13/2\sin t\,, \tag{3.65}$$
$$y = 1/2\,\mathrm{e}^{-3t}\sin 4t + 2\,\mathrm{e}^{-3t}\cos 4t + 8\,\sin t - 2\,\cos t\,. \tag{3.66}$$

The first two terms in each expression are the homogeneous solutions. Notice that the excitation is a sinusoid of frequency $\omega = 1$ and that in the steady state ($t \longrightarrow \infty$), the solution consists of only sinusoids of the same frequency.

3.8 Numerical solution

If the excitation to a circuit cannot be expressed in elementary functions, numerical evaluation of the convolution integral will be required. If numerical solution is desired, we might as well solve the state equations numerically in the first place. As we saw in the beginning of the chapter, the formulation of state equations is not an easy task. One must identify capacitor loops and inductor cutsets. If we are only interested in computing the time response, there is no need to derive the state equations. We should use a description by node equations as we did in the last chapter. Node equations can be directly set up from the topology of the circuit and can be solved numerically using the Backward Euler method. Once we have the node voltages, the state variables are determined.

3.9 Analog computer simulation

A state equation can also be solved on an analog computer. Consider a second order circuit described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{matrix} x_1 \\ x_2 \end{matrix}\Bigg] + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}\,, \tag{3.67}$$

with initial values $x_1(0)$ and $x_2(0)$. The equation can be implemented in an analog computer as shown in Fig. 3.12. The advantage of analog computer simulation is that the output $x_1(t)$ and $x_2(t)$ are obtained instantaneously, whereas in digital simulation the input must be sampled and the state equations be solved time point by time point. The output is not available until all its past values have been computed.

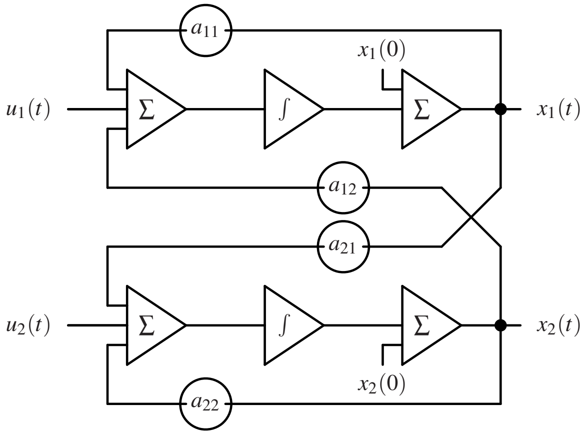

Fig. 3.12 Analog computer simulation of a system of two state equations

The disadvantages of analog computation are its limited accuracy and limited range of values of the variables.

3.10 Exponential excitation

There is an important class of excitations that requires special attention. It is one in which the excitation $\mathbf{u}(t) = \mathbf{U}e^{st}$, where $s = \sigma + j\omega$ is the complex frequency and $\mathbf{U}$ is a vector of constants, possibly complex. This class includes excitations of the form $\mathbf{u}(t) = \mathbf{U}\,e^{j\omega}$, $\mathbf{u}(t) = \mathbf{U}\cos\omega t$ and $\mathbf{u}(t) = \mathbf{U}\sin\omega t$ as special cases. We are interested in the response of the circuit in the steady state, namely, as $t \to \infty$.

Let the response $\mathbf{x}(t) = H(t)\mathbf{x}(0) + \mathbf{x}_p(t)$, where $\mathbf{x}_p(t)$ is a particular solution due to excitation $\mathbf{u}(t) = \mathbf{U}e^{st}$. Let us restrict ourselves to the practical case of circuits with finite losses so that the eigenvalues all have negative real parts. As $t \to \infty$, the response due to initial conditions all decay to zero, and $\mathbf{x}(t) = \mathbf{x}_p(t)$. Let $\mathbf{x}_p(t) = \mathbf{X}e^{st}$. Substituting into the inhomogeneous state equations, we find $\mathbf{X}$ is to be determined from:

$$(sI - A)\mathbf{X} = B\mathbf{U}\,, \tag{3.68}$$

which is a system of linear algebraic equations with s as a parameter. Each term of $\mathbf{X}$ will be a ratio of two polynomials in s, or a rational function. The denominator polynomial is in fact $\det(sI - A)$, or the characteristic polynomial of the state equations. So the poles of the rational functions are the eigenvalues of the circuit.

The steady state response is then

$$\mathbf{x}(t) = \mathbf{X}(s)e^{st}\,. \tag{3.69}$$

Since any circuit response is a linear combination of the state variables and excitation, we conclude

Theorem 3.5. *The voltages and currents in a circuit under excitations of the form* $\mathbf{u}(t) = \mathbf{U}e^{st}$ *are of the form* $\mathbf{y}(t) = \mathbf{Y}(s)e^{st}$ *in the steady state.*

To find the vector $\mathbf{Y}(s)$, we return to the state equations in normal form and by direct substitution we have

$$\mathbf{Y}(s) = C\mathbf{X}(s) + D\mathbf{U} = [C(sI - A)^{-1}B + D]\mathbf{U}. \tag{3.70}$$

The term inside the brackets plays the role of transforming a vector $\mathbf{U}$ of input to a vector of output $\mathbf{Y}(s)$ and it is called the *transfer function matrix*. Each term of the matrix is a rational function of s.[9]

Equation (3.70) gives the formal solution of a linear circuit in the steady state under exponential excitation. In the case the circuit is lossless, the response due to initial conditions does not decay to zero. Instead, it is a sum of sinusoids, as we saw in Example 3.3. The total response is the sum of the sinusoids plus the exponential response. For simplicity, when we speak of steady state response in a lossless circuit, we assume the initial conditions are all zero.

Problems

3.1. Write the state equations of a circuit consisting of an inductor L in parallel with a capacitor C. Obtain the solution in terms of the initial inductor current $i_L(0)$ and initial capacitor voltage $v_c(0)$. Show that the trajectory is an ellipse in the state space.

3.2. Show that the eigenvalues of the circuit in Fig. 3.13 are $-1 \pm j$. Find the complete solution for arbitrary initial conditions and arbitrary excitation $E(t)$. Let $C = 1\mathsf{F}$, $L = 1\mathsf{H}$, $R_1 = R_2 = 1\Omega$. Plot the trajectory of the homogeneous solution for two different sets of initial conditions in the state space.

3.3. In the circuit of Fig. 3.14, $C_1 = C_2 = C_3 = 1\mathsf{F}$, and $R_1 = R_2 = 1\Omega$. Show that the eigenvalues are -1 and $-1/3$. Let the excitation be $E(t) = 10\cos\omega t$. Find the steady state response.

3.4. In the circuit shown in Fig. 3.15, let $v_{out}(t)$ be the voltage across R_2 and $E(t) = 2\mathrm{e}^{-2t}$ for $t \geq 0$; $E(t) = 0$ otherwise. Show that

$$v_{out}(t) = \begin{cases} \mathrm{e}^{-t} - \frac{8}{9}\mathrm{e}^{-2t} - \frac{1}{9}\mathrm{e}^{-1/5t}, & \text{for } t \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

[9] In MATLAB®, $[num, den] = ss2tf(A,B,C,D,k)$, with an input component $U_k = 1$ and all others being zero, will return a vector of transfer functions whose numerator polynomials are in the vector *num* and whose denominator polynomial, being the same for all, is in *den*.

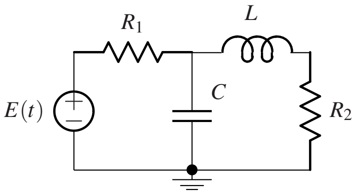

Fig. 3.13 A simple *RLC* circuit

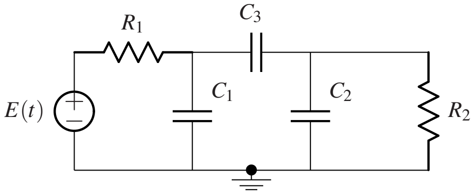

Fig. 3.14 A second order *RC* circuit.

The element values are: $L_1 = L_2 = 1\,\mathsf{H}$, $L_3 = 2\,\mathsf{H}$, $R_1 = R_2 = 1^{\cdot}\,\Omega$.

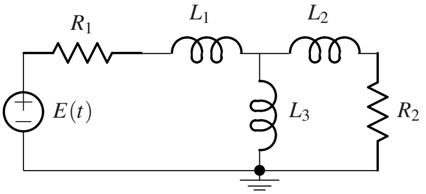

Fig. 3.15 A second order *RL* circuit

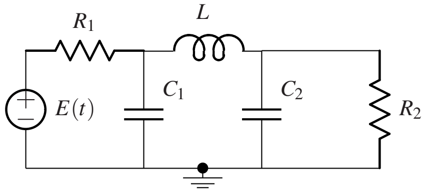

Fig. 3.16 A third order low-pass filter

3.5. The excitation $E(t)$ of the circuit of Fig. 3.16 is defined as: $E(t) = 1\,\mathsf{V}$ for $t \le 0$; and $E(t) = 0$ otherwise. Show that the voltage across R_2 for $t \ge 0$ is

$$v_2(t) = \frac{1}{2}\,\mathrm{e}^{-t} + \frac{\sqrt{3}}{3}\,\mathrm{e}^{-t/2}\sin\frac{\sqrt{3}}{2}t\,.$$

The element values are $R_1 = R_2 = 1\,\Omega$, $C_1 = C_2 = 1\,\mathsf{F}$, and $L = 2\,\mathsf{H}$. Plot the output $v_2(t)$ for $0 \le t \le 10s$.

3.6. Apply the Backward Euler method to solve the state equations of Problem 3.5 with $E(t) = \sin t + r(t)$, where $r(t)$ is random noise whose amplitude is uniformly distributed in the range $[-0.1, 0.1]$. Plot the output.

3.7. In the circuit of Fig. 3.17, suppose the initial voltage across C_1 is 1V, all the other initial conditions being zero. Show that the voltage across g_4 for all $t \ge 0$ is given by

$$v_{4n}(t) = 0.2225\,\mathrm{e}^{\alpha t}\cos\beta t - .0087\,\mathrm{e}^{\alpha t}\sin\beta t - 0.1434\,\mathrm{e}^{\lambda_3 t} - 0.0791\,\mathrm{e}^{\lambda_4 t}\ ,$$

where $\alpha = -0.5563$, $\beta = 0.9145$, $\lambda_3 = -1.1255$ and $\lambda_4 = -0.6786$. The element values are: $g_1 = 1\,\mathsf{S}$, $g_2 = 2\,\mathsf{S}$, $g_3 = 3\,\mathsf{S}$, $g_4 = 4\,\mathsf{S}$, $C_1 = C_2 = 1\,\mathsf{F}$, $L_1 = L_2 = 1\,\mathsf{H}$. Plot the projection of the state trajectory in the various 2-D planes to study the dynamics of the circuit.

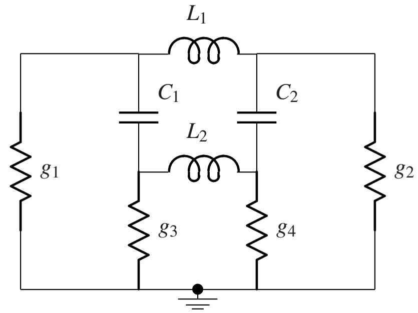

Fig. 3.17 A fourth order circuit

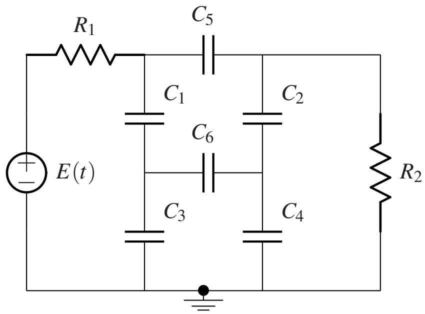

Fig. 3.18 A circuit with two zero eigenvalues

3.8. Show that in the circuit of Fig. 3.18 the voltage across R_2 is, to 4-digit accuracy,

$$v_2(t) = \int_0^t [0.4813\,e^{\lambda_1(t-\tau)} - 0.0440\,e^{\lambda_2(t-\tau)}]E(\tau)\,d\tau,$$

where $\lambda_1 = -0.9645$ and $\lambda_2 = -0.0882$. Assume the initial conditions are all zero. Let the input be a pulse $E(t) = \sin^2 \pi t/5$ for $0 \le t \le 5$; $E(t) = 0$, otherwise. Find $v_2(t)$ for $0 \le t \le 10$. Use numerical convolution and compare the solution with that obtained by backward Euler. The element values are $C_1 = 1$ F, $C_2 = 2$ F, $C_3 = 3$ F, $C_4 = 4$ F, $C_5 = 5$ F, $C_6 = 6$ F, and $R_1 = R_2 = 1\,\Omega$.

3.9. Show that the impulse response of a five-section RC ladder of Problem 2.13, which models a length of interconnect in an integrated circuit, with the output taken at the last node and $R = 1\,\Omega$ and $C = 0.1$ F, is given by:

$$v_{out}(t) = 0.554\,e^{\lambda_1 t} - 1.788\,e^{\lambda_2 t} + 2.720\,e^{\lambda_3 t} - 2.500\,e^{\lambda_4 t} + 1.014\,e^{\lambda_5 t} \tag{3.71}$$

where

$$\lambda_1 = -36.8250 \qquad \lambda_2 = -28.3083 \qquad \lambda_3 = -17.1537$$
$$\lambda_4 = -6.9028 \qquad \lambda_5 = -0.8101$$

Use the backward Euler method to compute the output voltage for an input pulse of height 1 V and duration of 1 second. The result should be as shown in Fig. 3.19, which also includes a plot of the impulse response.

3.10. Consider a fourth order LC circuit consisting of an inductor L_1 in series with a capacitor C_1 and with a parallel combination of inductor L_2 and capacitor C_2. Let $L_1 = 1$ H, $C_1 = 1/25$ F, $L_2 = 18$ H, and $C_2 = 1/72$ F. Let i_{L1}, v_{C1}, i_{L2} and v_{C2} be the state variables. Show that the responses to an initial voltage $v_{C1}(0) = 1$ V are:

$$i_{L1} = \frac{-16}{165}\sin 10t - \frac{1}{33}\sin t \qquad v_{C1} = \frac{25}{33}\cos t + \frac{8}{33}\cos 10t$$
$$i_{L2} = \frac{-4}{99}\sin t + \frac{2}{495}\sin 10t \qquad v_{C2} = \frac{8}{11}\cos 10t - \frac{8}{11}\cos t$$

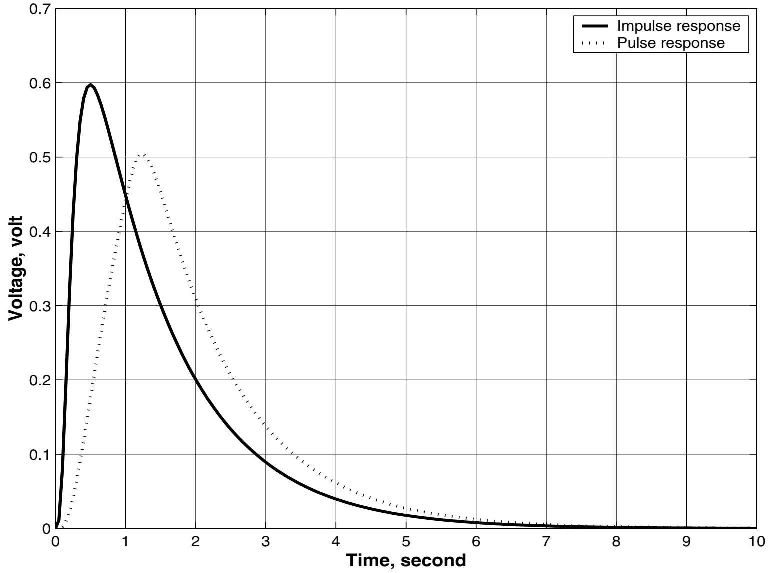

Fig. 3.19 Impulse response and pulse response of a five-section *RC* ladder that represents a length of interconnect in an integrated circuit (Problem 3.9)

Plot the trajectory projected on the $i_{L1} - v_{C1}$ plane, the $i_{L2} - v_{C1}$ plane, the $i_{L1} - v_{C2}$ plane, and the $i_{L2} - v_{C2}$ plane. The plots should be as shown in Fig. 3.20.

3.11. In the circuit of Fig. 3.21, the initial capacitor voltage $v_1(0) = 1\,\mathrm{V}$. All the other initial conditions are zero. At $t = 0$, switch S is closed. Show that the state variables for $t \geq 0$ are given by:

$$v_1(t) = \left(\frac{t^2}{3} + t + 1\right)\mathrm{e}^{-t}, \quad v_2(t) = \frac{4}{3}t^2\mathrm{e}^{-t}, \quad i_L(t) = \frac{8}{9}(t^2 + t)\mathrm{e}^{-t}.$$

Demonstrate that Tellegen's theorem holds.

3.12. The initial conditions of the circuit of Fig. 3.22 are all zero. Show that the capacitor voltage $v_c(t)$ for $t \geq 0$ is given by

$$v_c(t) = \int_0^t \left[\frac{2}{3}(t-\tau) - \frac{1}{6}(t-\tau)^2\right] \mathrm{e}^{-(t-\tau)/2} E(\tau)\,\mathrm{d}\tau.$$

3.13. Show that the transfer function $H(s) = V_c/E$ of the last problem, where $V_c\mathrm{e}^{st}$ is the response to $E\mathrm{e}^{st}$, is

$$H(s) = \frac{2/3\,s}{(s+1/2)^3}.$$

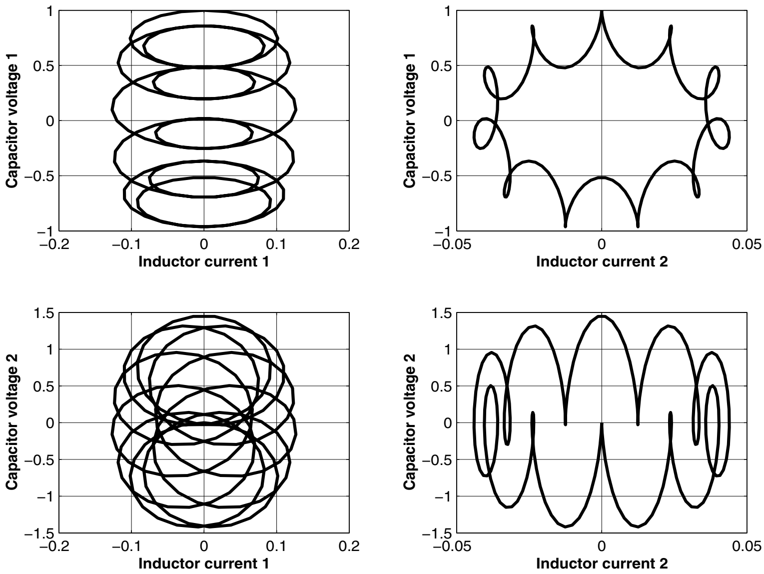

Fig. 3.20 Trajectory of a fourth order *LC* circuit projected on the various subspaces of the state space (Problem 3.10)

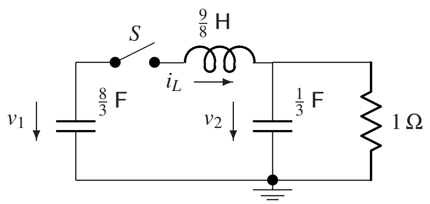

Fig. 3.21 Circuit with repeated eigenvalues of multiplicity three

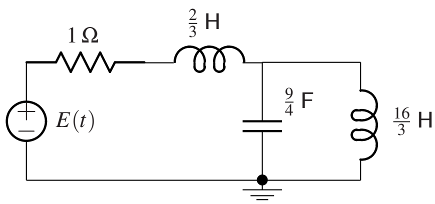

Fig. 3.22 Circuit with repeated eigenvalues of multiplicity three

3.14. Show that the transfer function of a five-section *RC* ladder of the type of Problem 3.9 with $R = 1\,\Omega$ and $C = 1\,\mathsf{F}$ is:

$$H(s) = \frac{V_{out}}{E} = \frac{1}{s^5 + 9s^4 + 28s^3 + 35s^2 + 15s + 1} \tag{3.72}$$

By frequency scaling, find the transfer function of a ladder with $R = 1\,\Omega$ and $C = 0.1\,\mathsf{F}$. Show that the poles of this transfer function equal in value to those of the eigenvalues of the state equations of the ladder.

Chapter 4
Properties in the Frequency Domain

In the last chapter, we learned that the steady state response to an exponential excitation of the form $u(t) = U\mathrm{e}^{st}$ is of the same form: $x(t) = X(s)\mathrm{e}^{st}$, where $X(s)$ is a vector of rational functions of the complex variable $s = \sigma + j\omega$. In this chapter, we will study the analytic properties of $X(s)$ that are important to the understanding of circuit behavior in the frequency domain. In particular we will derive relations between the real and imaginary parts, between the magnitude and phase, and between the phase and group delay, of a circuit function along the $j\omega$-axis of the complex plane. These relations help us to determine theoretical limitations of circuits.

4.1 Preliminaries

Consider an *RLC* circuit with an excitation $e(t) = E\mathrm{e}^{st}$. In the steady state, the element voltages and element currents will be of the form: $v_k(t) = V_k(s)\mathrm{e}^{st}$ and $i_k(t) = I_k(s)\mathrm{e}^{st}$, respectively. Since *KVL* and *KCL* equations are homogeneous, the complex voltages $V_k(s)$ satisfy *KVL* and the complex currents $I_k(s)$ *KCL*. As a result, Tellegen's theorem (Theorem 2.5) holds for the complex voltages and currents:

Theorem 4.1. *In the steady state under exponential excitation, the complex voltages and complex currents satisfy*

$$\sum_k V_k(s)\,I_k(s) = 0, \quad \sum_k V_k^*(s)\,I_k(s) = 0, \textit{and} \quad \sum_k V_k(s)\,I_k^*(s) = 0 \tag{4.1}$$

for any s, where $I_k^*(s)$ $(V_k^*(s))$ *is the complex conjugate of* $I_k(s)$ $(V_k(s))$.

We will have occasions to use this theorem in later chapters.

As to the definitions of elements, it follows from linearity that we have for a resistor: $V_R = RI_R$, an inductor: $V_L = sLI_L$, and a capacitor: $V_C = I_C/sC$. We define the *impedance* of an element as the ratio: V/I, and we have the impedance of a resistor is: $Z_R = R$, an inductor: $Z_L = sL$, and a capacitor: $Z_C = 1/sC$. Correspondingly, the *admittances*, being the ratio I/V, are respectively, $Y_G = 1/R = G$, $Y_L = 1/sL$, and

$Y_C = sC$. It is not possible to obtain similar relations for nonlinear elements. In fact impedance or admittance is not defined for them.

It is remarkable that nature provides three and only three kinds of elements with impedances that are constant, proportionate, and inversely proportionate to s. With these three kinds of elements, we are able to construct all kinds of circuits which make modern communications possible.

4.2 Modified node equations

In the frequency domain, circuits are best described by modified node equations, which are *KCL* equations written for all the non-ground nodes plus equations that describe the voltage sources and coupled inductors, if any. The variables are node voltages and the currents in the voltage sources and coupled inductors.

Consider the circuit shown in Fig. 4.1. Denote the node voltages by V_{1n}, V_{2n}, etc. Denote the currents in the voltage sources by I_{E1} and I_{E2}. Let each element be represented by its admittance. Then the modified node equations are:

$$\begin{bmatrix} y_1 & -y_1 & 0 & 0 & -1 & 0 \\ -y_1 & y_1+y_2 & 0 & -y_2 & 0 & -1 \\ 0 & 0 & y_3+y_5 & -y_3 & 1 & 0 \\ 0 & -y_2 & -y_3 & y_3+y_4+y_2 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1n} \\ V_{2n} \\ V_{3n} \\ V_{4n} \\ I_{E1} \\ I_{E2} \end{bmatrix} = \begin{bmatrix} J_1 \\ 0 \\ J_2 \\ 0 \\ E_1 \\ E_2 \end{bmatrix}. \tag{4.2}$$

In general, the node equations take the form:

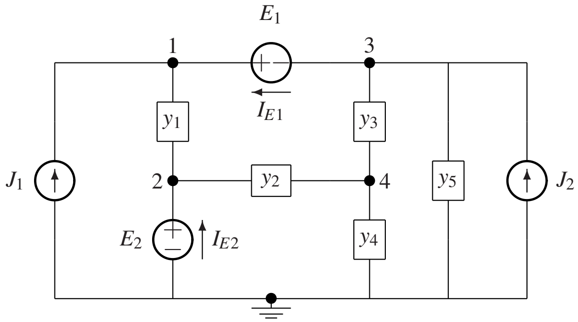

Fig. 4.1 A circuit to illustrate construction of modified node equations

$$\begin{bmatrix} Y & A \\ -A^T & 0 \end{bmatrix} \begin{bmatrix} V \\ I_E \end{bmatrix} = \begin{bmatrix} J \\ E \end{bmatrix}, \tag{4.3}$$

where $Y(s) = [y_{ij}(s)]$ is the admittance matrix and $A = [a_{ij}]$ is the connection matrix. They are constructed as follows.

1. y_{ii} is the sum of the admittances connected to node i;
2. $y_{ij} = y_{ji}$ is -1 times the admittance connected between nodes i and j;
3. If voltage source E_k is connected between nodes r and s with its "+" terminal at node r, then $a_{rk} = -1$ and $a_{sk} = 1$. (Note the polarity convention of the current.)

Further, V is the vector of node voltages, I_E is the vector of currents in the voltage sources, J is the vector of input current sources, and E is the vector of input voltage sources. The construction of the node equations is systematic. Note that the admittance matrix $Y(s)$ is symmetric.

4.3 Circuits with transconductances

Next, we consider a small-signal equivalent circuit under an exponential excitation of small amplitude. Since the circuit is linear, each element can be represented by its admittance and the circuit variables are all small-signal quantities, which obey *KVL* and *KCL*. The node equations are constructed as before except for transconductances.

Suppose in the circuit of Fig. 4.1 we replace y_3 by a transconductance whose current $i_{3,4} = g_m(V_{1n} - V_{2n})$, namely being controlled by the voltage across node 1 and node 2. The modified node equations becomes

$$\begin{bmatrix} y_1 & -y_1 & 0 & 0 & -1 & 0 \\ -y_1 & y_1+y_2 & 0 & -y_2 & 0 & -1 \\ g_m & -g_m & y_3+y_5 & -y_3 & 1 & 0 \\ -g_m & -y_2+g_m & -y_3 & y_3+y_4+y_2 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1n} \\ V_{2n} \\ V_{3n} \\ V_{4n} \\ I_{E1} \\ I_{E2} \end{bmatrix} = \begin{bmatrix} J_1 \\ 0 \\ J_2 \\ 0 \\ E_1 \\ E_2 \end{bmatrix}. \tag{4.4}$$

In general, if a transconductance is connected across nodes r and s whose current is controlled by the voltage across nodes p and q, then add g_m to y_{rp} and y_{sq} and add $-g_m$ to y_{rq} and y_{sp}. Note that the admittance matrix is not symmetric and we have

Observation 4.1 *The admittance matrix of the small-signal equivalent of an* MOS *or bipolar transistor circuit in general is not symmetric.*

4.4 Reciprocity

Returning to an *RLC* circuit, let us apply a current source at node 1 and compute the voltage at node 2. Let the node equation be

$$YV_n = \mathbf{e}_1 J\,, \tag{4.5}$$

where $\mathbf{e}_k$ is the unit vector k, which is a vector of zeros except that the kth element is unity. J is the source current. Formally, node voltage V_{2n} is given by

$$V_{2n} = \frac{\Delta_{1,2}}{\Delta} J\,, \tag{4.6}$$

where $\Delta_{1,2}$ is the $1-2$ cofactor of the determinant Δ of Y. Now let us place J at node 2 and compute node voltage V'_{1n}. Let the node equation be

$$YV'_n = \mathbf{e}_2 J\,. \tag{4.7}$$

Node voltage V'_{1n} is

$$V'_{1n} = \frac{\Delta_{2,1}}{\Delta} J\,. \tag{4.8}$$

Since Y is symmetric, $\Delta_{i,j} = \Delta_{j,i}$ and $V_{2n} = V'_{1n}$. We define a *reciprocal circuit* as one in which the node voltage at node p due to a current applied at node q is equal to the node voltage at q if the same current is removed and applied at node p. We state the observation in a theorem:

Theorem 4.2 (Reciprocity). *An* RLC *circuit is a reciprocal circuit.*

In general, a circuit containing transconductances is not reciprocal. This is one of the major differences between an *RLC* circuit and the small-signal equivalent of an electronic circuit.

Reciprocity can also be defined in terms of a voltage source. Let us compute the current in a short-circuit connected between nodes p and q due to a voltage source connected across nodes i and j. Now exchange the voltage source with the short-circuit and compute the current in the short-circuit between i and j. The two currents are equal. The proof is left as an exercise.

4.5 Impedance, admittance

Let $\mathscr{N}$ be an *RLC* circuit with n nodes. Apply a current source to node 1. Let the node equation be

$$Y(s)V_n = \mathbf{e}_1 J\,. \tag{4.9}$$

Each term of $Y(s)$ is of the form $g + sc + 1/s\ell$. The node voltage at node 1 is

$$V_{1n} = \frac{\Delta_{11}(s)}{\Delta(s)} J\,. \tag{4.10}$$

We define the *driving-point impedance* or simply *input impedance* across node 1 and ground as

$$Z(s) = \frac{Y_{1n}}{J} = \frac{\Delta_{11}(s)}{\Delta(s)} = \frac{P(s)}{Q(s)}\,, \tag{4.11}$$

where $P(s)$ and $Q(s)$ are the numerator and denominator polynomials, respectively, whose coefficients are all real. So we have

Observation 4.2 *$Z(s)$ is real when s is real.*

Observation 4.3 *The poles and zeros of $Z(s)$ are real or occur in complex conjugate pairs.*

The circuit $\mathscr{N}$ may also be excited by a voltage source E connected across node 1 and ground. The modified node equation is

$$\begin{bmatrix} Y & -e_1 \\ e_1^T & 0 \end{bmatrix} \begin{bmatrix} V_n \\ I_E \end{bmatrix} = e_n E\,. \tag{4.12}$$

Let the modified node matrix be Y'. Solving for the source current I_E and dividing by E, we get the *driving-point* or *input admittance* across node 1 and ground given by

$$Y(s) = \frac{I_E}{E} = \frac{\Delta'_{nn}(s)}{\Delta'(s)}\,, \tag{4.13}$$

where $\Delta'(s)$ is the determinant of Y' and Δ'_{nn} its $n-n$ cofactor. It can be shown that

$$\Delta'_{nn}(s) = \Delta(s)\,, \qquad \Delta'(s) = \Delta_{11}(s)\,, \tag{4.14}$$

and

$$Y(s) = \frac{\Delta(s)}{\Delta_{11}(s)} = \frac{1}{Z(s)} = \frac{Q(s)}{P(s)}\,, \tag{4.15}$$

as expected.

4.5.1 Poles and zeros

Since $Z(s)$ and $Y(s)$ are both response functions, according to Chapter 3, their poles have non-positive real parts, and we have

Observation 4.4 *The poles and zeros of an input impedance (admittance) of an* RLC *circuit all lie to the left or on the $j\omega$ axis of the s-plane.*

We define a *Hurwitz polynomial* as a polynomial with real coefficients whose zeros all have non-positive real parts, and a *strictly Hurwitz polynomial* as one whose zeros all have negative real parts [34, 28]. We then have the following.

Observation 4.5 *The input impedance (admittance) function of an* RLC *circuit is a ratio of two Hurwitz polynomials.*

Observation 4.6 *The impedance or admittance function of an* RLC *circuit is analytic in the right-half of the s-plane.*

4.5.2 Real and imaginary parts

Next, let us examine the properties along the $j\omega$-axis, namely, we want to look at the frequency response of circuits. Let an impedance function along the $j\omega$-axis be:

$$Z(j\omega) = R(\omega) + jX(\omega), \tag{4.16}$$

where $R(\omega)$ is the real part and $X(\omega)$ the imaginary part. Next consider the complex conjugate of $Z(j\omega)$:

$$Z^*(j\omega) = Z(-j\omega) = R(\omega) - jX(\omega). \tag{4.17}$$

On the other hand,

$$Z(-j\omega) = Z(j(-\omega)) = R(-\omega) + jX(-\omega). \tag{4.18}$$

So we get an important result:

$$R(\omega) = R(-\omega), \qquad X(\omega) = -X(-\omega), \tag{4.19}$$

and we have:

Theorem 4.3. *The real part of an impedance function along the $j\omega$ axis is an even function of ω and the imaginary part is an odd function of ω.*

To obtain the real (and imaginary) part from $Z(j\omega)$, it is convenient to write

$$Z(s) = \frac{m_1 + n_1}{m_2 + n_2}, \tag{4.20}$$

where m_1 and n_1 are the even and odd parts of the numerator polynomial, respectively, and m_2 and n_2 are the even and odd parts of the denominator polynomial. Then we have

$$R(\omega) = \left. \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2} \right|_{s=j\omega}. \tag{4.21}$$

We will have occasions in latter chapters to use this formula.

4.5.3 Impedance function from its real part

It is possible to find an impedance function whose real part is specified as a rational function in ω. From Eq. (4.16) and (4.17),

$$R(\omega) = \frac{1}{2}\left[Z(j\omega) + Z(-j\omega)\right], \tag{4.22}$$

which is to say

$$R(s/j)|_{s=j\omega} = \frac{1}{2}\left[Z(s)+Z(-s)\right]\bigg|_{s=j\omega}. \tag{4.23}$$

The procedure is to replace ω by s/j in $R(\omega)$ to obtain a rational function in s^2. Expand the rational function in partial fractions with one part having poles in the left-half of the s-plane and the other part having poles in the right-half s-plane. Twice the first part is the impedance function $Z(s)$ whose real part is $R(\omega)$. The impedance function thus obtained is not unique since we can add any odd function of s to $Z(s)$ and the new impedance function has the same real part.

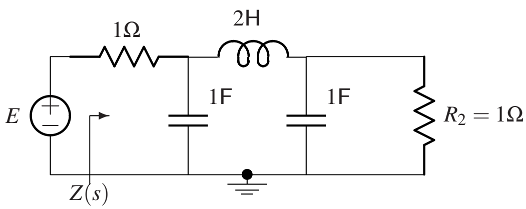

Fig. 4.2 A third order low-pass filter

As an example, the input impedance of the circuit of Fig. 4.2 is

$$Z(s) = \frac{2s^3+4s^2+4s+2}{2s^3+2s^2+2s+1}. \tag{4.24}$$

Its real part along the $j\omega$-axis is

$$R(\omega) = \frac{4\omega^6-4\omega^4+2}{4\omega^6-4\omega^4+1}. \tag{4.25}$$

Suppose $R(\omega)$ is given. Let us find $Z(s)$. Following the procedure outlined above, we obtain [1]

$$\begin{aligned} R(\omega) &= \frac{4\omega^6-4\omega^4+2}{4\omega^6-4\omega^4+1} = \frac{-4s^6-4s^4+2}{-4s^6-4s^4+1}\bigg|_{s=j\omega} \\ &= \left[\frac{1}{2}+\frac{r_1}{s+p_1}+\frac{r_2}{s+p_2}+\frac{r_2^*}{s+p_2^*}+\frac{1}{2}+\frac{-r_1}{s-p_1}+\frac{-r_2}{s-p_2}+\frac{-r_2^*}{s-p_2^*}\right]_{s=j\omega}, \end{aligned} \tag{4.26}$$

where the residues are: $r_1 = 0.1411$, $r_2 = 0.1795 + j0.0898$ and the poles are: $p_1 = 0.6478$, $p_2 = 0.1761 + j0.8607$. The first half has poles in the left-half s-plane and is associated with $Z(s)$. Assembling the terms and multiplying by 2, we get as before

[1] In MATLAB®, $[r, p, k] = residue(b, a)$ obtains the partial fraction expansion of rational function b/a where a and b are expressed as vectors of coefficients of polynomials a and b. The residues are placed in vector r and the poles in vector p. Vector k contains the coefficients of any excess polynomial.

$$Z(s) = \frac{2s^3 + 4s^2 + 4s + 2}{2s^3 + 2s^2 + 2s + 1}\,. \tag{4.27}$$

4.6 Transfer function

A transfer function relates the output to the input. Consider an *RLC* circuit with an input voltage source connected across node 1 and ground. Take the output voltage at node 2. Let the node equation be

$$\begin{bmatrix} Y & -e_1 \\ e_1^T & 0 \end{bmatrix} \begin{bmatrix} V_n \\ I_E \end{bmatrix} = e_n E\,. \tag{4.28}$$

Solving for V_{2n} we find the transfer function $F(s)$ to be

$$F(s) = \frac{V_{2n}}{E} = \frac{\Delta'_{1,2}}{\Delta'} = \frac{P(s)}{Q(s)}\,, \tag{4.29}$$

which is a ratio of two polynomials with real coefficients. Its poles and zeros are real or occur in complex conjugate pairs. Since $F(s)$ is a response function, its poles must have non-positive real parts. So

Observation 4.7 *The transfer function of an* RLC *circuit is analytic in the right-half s-plane.*

Note that the poles of $F(s)$ are the same as the zeros of the driving point impedance seen by the voltage source. See Eqs. (4.11) and (4.14).

Along the $j\omega$-axis, we often write

$$F(j\omega) = M(\omega)\,\mathrm{e}^{j\theta(\omega)} \tag{4.30}$$

where $M(\omega)$ is the magnitude function and $\theta(\omega)$ is the phase function, given by:

$$M(\omega) = \sqrt{F(j\omega)F(-j\omega)}\,, \tag{4.31}$$

$$\theta(\omega) = \arctan\frac{\Im\{F(j\omega)\}}{\Re\{F(j\omega)\}} \pm 2n\pi\,. \tag{4.32}$$

An integer multiple of $\pm 2\pi$ is added to the phase function to make it a continuous function of ω. From these expressions, we see

Observation 4.8 *The magnitude function of a transfer function is an even function of ω and the phase function is odd.*

In practice, we speak of the gain $G(\omega)$ and phase of a transfer function as:

$$G(\omega) = \Re\{\ln F(j\omega)\} = \ln M(\omega)\,, \tag{4.33}$$

$$\theta(\omega) = \Im\{\ln F(j\omega)\}\,, \tag{4.34}$$

and the gain in dB as

$$G_{dB}(\omega) = 20\log M(\omega)\,. \tag{4.35}$$

4.6.1 Frequency response

The frequency response of a circuit can be displayed in a Bode plot [8], which shows the gain $G_{dB}(\omega)$ and phase $\theta(\omega)$ on a logarithmic frequency scale. As an example, consider the circuit of Fig. 4.2. Take the output across R_2. The transfer function is found to be

$$F(s) = \frac{0.5}{s^3 + 2s^2 + 2s + 1}\,, \tag{4.36}$$

which has all its poles in the left-half s-planes. Its zeros are all at $s = \infty$. Its magnitude and phase functions are:

$$M(\omega) = \frac{0.5}{\sqrt{1 + \omega^6}}\,, \tag{4.37}$$

$$G_{dB}(\omega) = -6.02 - 10\log(1 + \omega^6)\,, \tag{4.38}$$

$$\theta(\omega) = -\arctan\frac{2\omega - \omega^3}{1 - 2\omega^2}\,. \tag{4.39}$$

These are plotted in Fig. 4.3.[2] At low frequencies, the dB-gain is approximately constant at -6 dB. At high frequencies, it decreases at a constant rate of -60 dB per decade on a logarithmic frequency scale.

As a second example, let us obtain the frequency response of the transfer function V_{3n}/E of the bandpass filter of Fig. 4.4. Writing the modified node equations with node voltages, inductor currents and source current as variables, we get

$$\begin{bmatrix} G_1 & -G_1 & 0 & 0 & 0 & -1 \\ -G_1 & G_1 + sC_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & G_2 + sC_2 & 0 & 1 & 0 \\ 0 & -1 & 0 & sL_1 & sM & 0 \\ 0 & 0 & -1 & sM & sL_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1n} \\ V_{2n} \\ V_{3n} \\ I_1 \\ I_2 \\ I_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E \end{bmatrix}. \tag{4.40}$$

Let the element values be: $G_1 = G_2 = 1/10\mathsf{S}$, $C_1 = C_2 = 1\mathsf{F}$, $L_1 = L_2 = 1\mathsf{H}$, and $M = 1/10\mathsf{H}$. The transfer function is found to be

$$F(s) = \frac{V_{3n}}{E} = \frac{100s}{9900s^4 + 1980s^3 + 20099s^2 + 2000s + 10000}\,, \tag{4.41}$$

---

[2] In MATLAB®, the function [mag, phs, w]=bode(n, d, w) places the magnitude in vector *mag* and phase in vector *phs* at each frequency in vector *w* of a transfer function whose numerator polynomial is *n* and denominator polynomial is *d*.

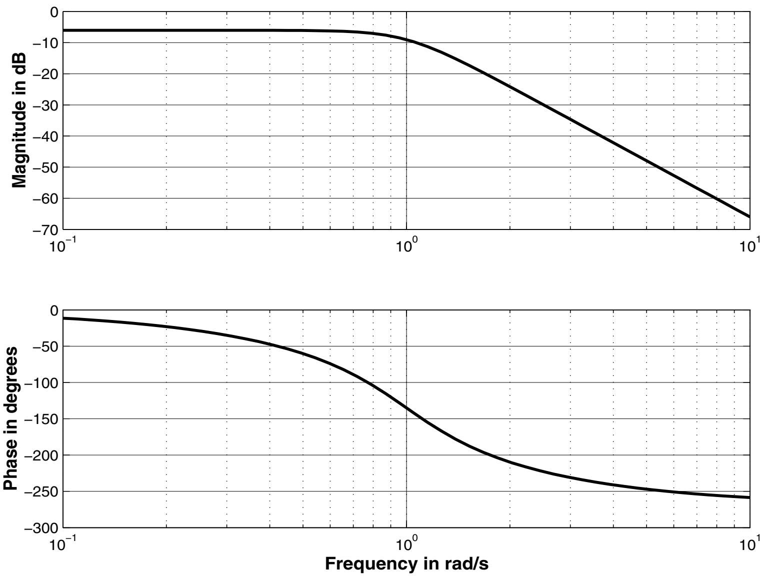

Fig. 4.3 Bode plot of transfer function of a low pass filter

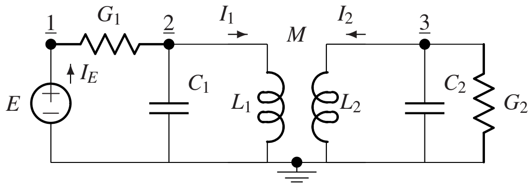

Fig. 4.4 A simple bandpass filter used in early radio receivers

which has poles at $-0.0500 \pm j1.0529$ and $-0.0500 \pm j0.9522$ and a zero at 0. The magnitude, gain in dB, and phase are shown in Fig. 4.5. In practice, the element values are not simple rational numbers and it is not advisable to obtain a symbolic solution of the node equations. If we are interested only in the frequency response, a numerical solution is preferred and the node equations are solved over the range of frequencies of interest, one by one.

4.6.2 Transfer function from its magnitude

There are occasions in filter synthesis where we need to find a transfer function $F(s)$ whose magnitude-squared function is given as some function of frequency. We begin with the observation that

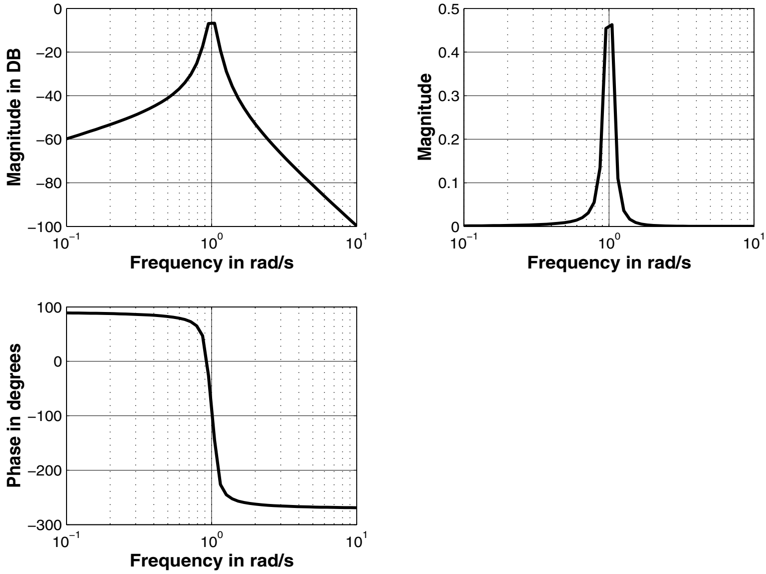

Fig. 4.5 Frequency response of a band-pass filter used in a radio receiver

$$M^2(\omega) = F(s)F(-s)|_{s=j\omega} \, . \tag{4.42}$$

This suggests that we replace ω with s/j in $M^2(\omega)$ to obtain a rational function in s^2, factor the numerator and denominator, and assign poles with negative real parts to $F(s)$. The zeros can be assigned to either $F(s)$ or $F(-s)$. As an example, consider the magnitude-squared function of Eq. (4.37). It can be factored as

$$M^2(s/j) = \frac{1}{2}\,\frac{1}{(s+1)(s^2+s+1)}\,\frac{1}{2}\,\frac{1}{(1-s)(1-s+s^2)} \, . \tag{4.43}$$

Picking factors with poles in the left-half s-plane, we recover the transfer function as before:

$$F(s) = \frac{1}{2}\,\frac{1}{(s+1)(s^2+s+1)} = \frac{0.5}{s^3+2s^2+2s+1} \, . \tag{4.44}$$

4.6.3 All-pass and minimum phase transfer functions

It is possible to construct a transfer function whose magnitude is constant for all ω. Such a function is called an *all-pass* transfer function and it consists of pole-zero pairs equidistant from the $j\omega$-axis. Since the poles must be in the left-half of the s-plane, the zeros are all in the right-half s-plane. It takes a general form:

$$F(s) = A \prod_{j,k} \frac{(s-a_k)}{(s+a_k)} \frac{(s^2 - a_j s + b_j)}{(s^2 + a_j s + b_j)} . \tag{4.45}$$

Now consider a transfer function $F(s)$ which has a zero (or a complex conjugate pair of zeros) in the right-half s-plane. We can write

$$\begin{aligned} F(s) &= \frac{P(s)}{Q(s)} = \frac{P_1(s)(s-a)}{Q(s)} = \frac{P_1(s)(s+a)}{Q(s)} \frac{(s-a)}{(s+a)} \\ &= F'(s) \frac{(s-a)}{(s+a)} = F'(s)F''(s) . \end{aligned} \tag{4.46}$$

Noting $F''(s)$ is an all-pass transfer function, we have

$$F(j\omega) = M(\omega)\mathrm{e}^{j\theta(\omega)} = M'(\omega)\mathrm{e}^{j\theta'(\omega)}\mathrm{e}^{j\theta''(\omega)} . \tag{4.47}$$

More explicitly

$$M(\omega) = M'(\omega) \qquad \theta(\omega) = \theta'(\omega) + \theta''(\omega) . \tag{4.48}$$

In words, given a transfer function $F(s)$ which has a zero in the right-half plane, there exists another transfer function $F'(s)$ which has the same magnitude function as $F(s)$ but its phase is less than the phase of $F(s)$ by the phase of an all-pass function.

We call a transfer function whose zeros are all in the left-half s plane a *minimum phase* transfer function, meaning that there does not exist another transfer function with the same magnitude function and with less phase.

4.6.4 Linear phase and group delay

Consider a filter, say a bandpass filter, with a transfer function $F(j\omega) = M(\omega)\mathrm{e}^{j\theta(\omega)}$. Let the center frequency of the passband be ω_o. Suppose the input signal consists of two (or more) frequency components ω_1 and ω_2, both in the passband. If the filter is well-designed, $M(\omega)$ will be relatively constant over the passband and $M(\omega_1) \approx M(\omega_2) \approx M(\omega_o)$. At the output, we want the signal to be as close to the input as possible.

Let the input and output signals be $E(t)$ and $V(t)$, respectively, written as:

$$E(t) = A_1\mathrm{e}^{j\omega_1 t} + A_2\mathrm{e}^{j\omega_2 t} \tag{4.49}$$

$$\begin{aligned} V(t) &= M(\omega_1)A_1\,\mathrm{e}^{j(\omega_1 t + \theta(\omega_1))} + M(\omega_2)A_2\,\mathrm{e}^{j(\omega_2 t + \theta(\omega_2))} \\ &\approx M(\omega_o)\left[A_1\,\mathrm{e}^{j\omega_1(t+\theta(\omega_1)/\omega)} + A_2\,\mathrm{e}^{j\omega_2(t+\theta(\omega_2)/\omega_2)}\right] \end{aligned} \tag{4.50}$$

So but for the delay $\theta(\omega_1)/\omega_1$ and $\theta(\omega_2)/\omega_2$, the output would be the same as the input multiplied by a constant. If the filter is such that its phase is a linear function of frequency:

$$\theta(\omega) = -\tau\omega \tag{4.51}$$

then the two components will have the same delay and the output will be:

$$V(t)|_{linear\,phase} = M(\omega_o)\left[A_1\,\mathrm{e}^{j\omega_1(t-\tau)} + A_2\,\mathrm{e}^{j\omega_2(t-\tau)}\right] = M(\omega_o)E(t-\tau) \tag{4.52}$$

namely, the output is a delayed, scaled replica of the input, which would be acceptable in most communication systems. However, the phase in general is not a linear function of frequency, and the output will have delay distortion. To accommodate a continuum of frequency components, we define a term, *group delay* $D(\omega)$:

$$D(\omega) = -\frac{\mathrm{d}\theta}{\mathrm{d}\omega} \tag{4.53}$$

as a measure of the delay of a signal with incremental frequency component $\mathrm{d}\omega$. If the phase is linear over the passband, the group delay is constant in the passband.

The group delay of a transfer function can be expressed in terms of its poles and zeros. Let the transfer function be:

$$F(s) = K\prod_k \frac{(s+z_k)}{(s+p_k)}\,\frac{(s^2+a_k s+b_k)}{(s^2+c_k s+d_k)}\,. \tag{4.54}$$

Then the group delay is given by:

$$D(\omega) = \sum_k \left\{ -\frac{z_k}{z_k^2+\omega^2} + \frac{p_k}{p_k^2+\omega^2} - \frac{a_k(b_k+\omega^2)}{(b_k-\omega^2)^2+a_k^2\,\omega^2} + \frac{c_k(d_k+\omega^2)}{(d_k-\omega^2)^2+c_k^2\,\omega^2}\,.\right\} \tag{4.55}$$

It is an even rational function of ω^2. In large circuits with realistic element values, it is not practical to find the zeros and poles of a transfer function, and it is not possible to obtain the phase function explicitly. If only numerical values of the delay is desired, there is a simple method to compute the delay without taking derivatives. This is shown in Sect. 4.9.2.

4.7 Relation between real and imaginary parts

That an impedance function or a transfer function can be recovered from its real part or magnitude along the $j\omega$-axis indicates that there is an analytic relation between the real part and imaginary part of a circuit function. This is indeed the case. In fact there are numerous such relations and we shall present a few here.

We need a theorem from the theory of functions of a complex variable [28].

Theorem 4.4. *Let $F(s)$ be analytic in a region $\mathscr{R}$ and on its boundary $\mathscr{C}$. Then*

$$\oint_{\mathscr{C}} F(s)\,\mathrm{d}s = 0\,. \tag{4.56}$$

This theorem is the starting point for the derivation of relations between the real and imaginary parts of a circuit function (impedance or transfer function), which is analytic in the right-half s-plane bounded by the $j\omega$-axis and the arc of a semicircle of infinite radius.

Theorem 4.5. *Let $F(s)$ be analytic in the right-half s-plane and on its boundary. Let*

$$F(j\omega) = R(\omega) + jX(\omega)\,. \tag{4.57}$$

Then

$$X(\omega) = \frac{2\omega}{\pi}\int_0^\infty \frac{R(\Omega)\,\mathrm{d}\Omega}{\Omega^2-\omega^2}\,, \tag{4.58}$$

$$R(\omega) = R(\infty) - \frac{2}{\pi}\int_0^\infty \frac{\Omega X(\Omega)\,\mathrm{d}\Omega}{\Omega^2-\omega^2}\,. \tag{4.59}$$

Proof. Let

$$G(s) = \frac{F(s)}{s^2+\omega_o^2} \tag{4.60}$$

$G(s)$ is analytic in the right-half s-plane except for two poles at $\pm j\omega_o$. Let $\mathscr{C}$ be a contour consisting of sub-contours $\mathscr{C}_1$, which extends from $s=-jR$ to $s=-j\omega_o-jr$, where r is the radius of a small semicircle around the pole at $-j\omega_o$ and R is the radius of a semicircle enclosing the right-half s plane; $\mathscr{C}_2$, which extends from $-j\omega_o+jr$ to $j\omega_o-jr$; $\mathscr{C}_3$, which extends from $j\omega_o+jr$ to jR; $\mathscr{C}_4$, which is the arc of a small semicircle around $-j\omega_o$; $\mathscr{C}_5$, which is the arc of a small semicircle around $j\omega_o$; and $\mathscr{C}_6$, which is the arc of a semicircle of radius R enclosing the right-half plane. These are shown in Fig. 4.6. As $R\to\infty$ and $r\to 0$, the contour encloses the entire right-half s plane. By Theorem 4.4, we have

$$\begin{aligned}\oint_{\mathscr{C}} G(s)\,\mathrm{d}s = \int_{\mathscr{C}_1} G(s)\,\mathrm{d}s + \int_{\mathscr{C}_2} G(s)\,\mathrm{d}s + \int_{\mathscr{C}_3} G(s)\,\mathrm{d}s +\\ \int_{\mathscr{C}_4} G(s)\,\mathrm{d}s + \int_{\mathscr{C}_5} G(s)\,\mathrm{d}s + \int_{\mathscr{C}_6} G(s)\,\mathrm{d}s = 0\,.\end{aligned} \tag{4.61}$$

Let I_k denote the integral along $\mathscr{C}_k$. Along $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$, we have

$$I_1 = \int_{-R}^{-\omega_o-r} \frac{[jR(\omega)-X(\omega)]\,\mathrm{d}\omega}{-\omega^2+\omega_o^2}\,, \tag{4.62}$$

$$I_2 = \int_{-\omega_o+r}^{\omega_o-r} \frac{[jR(\omega)-X(\omega)]\,\mathrm{d}\omega}{-\omega^2+\omega_o^2}\,, \tag{4.63}$$

$$I_3 = \int_{\omega_o+r}^{R} \frac{[jR(\omega)-X(\omega)]\,\mathrm{d}\omega}{-\omega^2+\omega_o^2}\,. \tag{4.64}$$

Along $\mathscr{C}_4$ and $\mathscr{C}_5$, we have

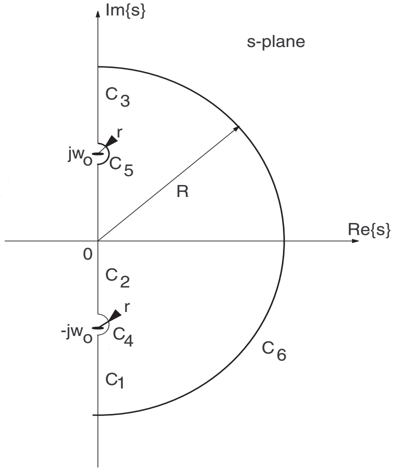

Fig. 4.6 The right-half s plane and its boundary segments

$$\lim_{r\to 0} I_4 = \lim_{r\to 0}\int_{-\pi/2}^{\pi/2} \frac{F(-j\omega_o + re^{j\theta})\,d\theta}{-2\omega_o - jre^{j\theta}} = \pi\frac{-R(\omega_o)+jX(\omega_o)}{2\omega_o}, \tag{4.65}$$

$$\lim_{r\to 0} I_5 = \pi\frac{R(\omega_o)+jX(\omega_o)}{2\omega_o}. \tag{4.66}$$

Lastly, along $\mathscr{C}_6$, $s = Re^{j\theta}$ and in the limit as $R \to \infty$,

$$\lim_{R\to\infty} I_6 = \lim_{R\to\infty}\int_{\pi/2}^{-\pi/2} \frac{F(Re^{j\theta})Re^{j\theta}j\,d\theta}{R^2e^{j2\theta}+\omega_o^2} = 0. \tag{4.67}$$

Adding and noting $R(\omega)$ is even and $X(\omega)$ is odd, we get

$$X(\omega_o) = \frac{2\omega_o}{\pi}\int_0^{\infty}\frac{R(\omega)\,d\omega}{\omega^2-\omega_o^2}. \tag{4.68}$$

After changing variables we obtain relation (4.58). The second relation is proved in the same way except that we use

$$G(s) = \frac{sF(s)}{s^2+\omega_o^2}. \tag{4.69}$$

□

The significance of relations (4.58) and (4.59) is that the real and imaginary parts of a function which is analytic in the right-half s-plane and along its boundary, are related. If one is known along the $j\omega$-axis, the other at any ω can be determined. There are many such relations [8]. Here is another pair, known as Hilbert transform pair: [28].

$$X(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R(\Omega)}{\Omega - \omega} \, \mathrm{d}\Omega \,, \tag{4.70}$$

$$R(\omega) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{X(\Omega)}{\Omega - \omega} \, \mathrm{d}\Omega \,, \tag{4.71}$$

Another direct application of Theorem 4.4 is the following.

Theorem 4.6. *Let $F(s)$ be analytic in the right-half plane and along its boundary. Suppose that it has an expansion about $s = \infty$ as follows.*

$$F(s) = F(\infty) + \frac{a_1}{s} + \frac{a_2}{s^2} + \dots . \tag{4.72}$$

Let $F(j\omega) = R(\omega) + jX(\omega)$. Then

$$\int_0^{\infty} [R(\omega) - R(\infty)] \, \mathrm{d}\omega = \frac{\pi}{2} a_1 \,. \tag{4.73}$$

The theorem says that the area under the real part curve above $R(\infty)$ is fixed by the first coefficient in the expansion of $F(s)$ about infinity [8]. See Problem 4.7.

4.8 Gain and phase relation

Starting with Eq. (4.70), one can derive the following relation (see Problem 4.9):

$$X(\omega) = \frac{1}{\pi} \int_0^{\infty} \frac{\mathrm{d}R(\Omega)}{\mathrm{d}\Omega} \ln \left| \frac{\Omega + \omega}{\Omega - \omega} \right| \mathrm{d}\Omega \,, \tag{4.74}$$

and an important theorem on gain and phase, given below.

Theorem 4.7. *Let $F(s)$ be a minimum phase transfer function. Let*

$$\ln F(j\omega) = G(\omega) + j\theta(\omega). \tag{4.75}$$

Further, define new frequency variables: $p = \log \omega$ and $P = \log \Omega$. Then

$$\theta(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}G(P)}{\mathrm{d}P} \right) \ln \left| \frac{\Omega + \omega}{\Omega - \omega} \right| \mathrm{d}P. \tag{4.76}$$

The logarithm term resembles an impulse, as shown in Fig. 4.7. The area under the curve is $\pi^2/2$ [8] and we have the following result and observation.

$$\theta(p) \approx \frac{\pi}{2} \left. \frac{\mathrm{d}G(P)}{\mathrm{d}P} \right|_{P=p} , \tag{4.77}$$

Observation 4.9 *The phase of a minimum phase transfer function at a frequency ω is $\pi/2$ times the slope of its dB gain at ω on a logarithmic frequency scale.*

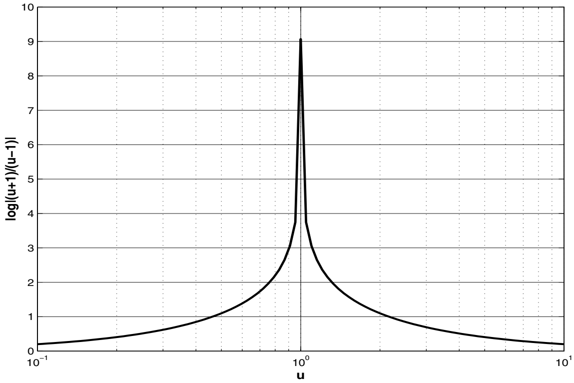

Fig. 4.7 The impulse-like function $\ln|(u+1)/(u-1)|$. The area under the curve is $\pi^2/2$

This can be seen in Figs. 4.3 and 4.5.

A tighter result can be obtained if we approximate the gain function $G(\omega)$ as a piece-wise linear function over the frequency range as shown in Fig. (4.8). Returning to the linear Ω scale, we write

$$\theta(\omega) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}G}{\mathrm{d}\Omega} \ln\left|\frac{\Omega+\omega}{\Omega-\omega}\right| \mathrm{d}\Omega\,. \tag{4.78}$$

Let the slope of the kth straight line segment be

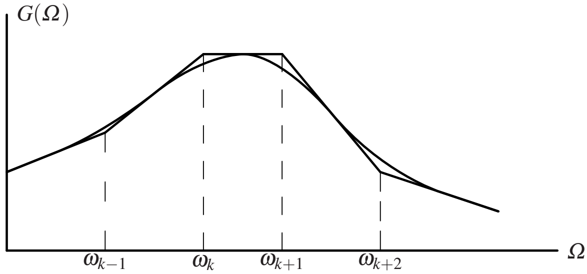

Fig. 4.8 Piece-wise linear approximation of a gain function

$$M_k = \frac{G(\omega_k) - G(\omega_{k-1})}{\omega_k - \omega_{k-1}}\,. \tag{4.79}$$

Let there be K segments, with K sufficiently large to cover all frequencies where the slope is significant. Then the phase at ω is [8]

$$\theta(\omega) = \frac{1}{\pi}\sum_{k=1}^{K} M_k[Q(\omega,\omega_k) - Q(\omega,\omega_{k-1}],\tag{4.80}$$

where

$$\begin{aligned} Q(\omega,\omega_k) &= \int_0^{\omega_k} \ln\left|\frac{\Omega+\omega}{\Omega-\omega}\right| \mathrm{d}\Omega \\ &= (\omega+\omega_k)\ln(\omega+\omega_k) + (\omega-\omega_k)\ln|\omega-\omega_k| - 2\omega\ln|\omega|\,. \end{aligned}\tag{4.81}$$

More explicitly, we have

$$\begin{aligned} \theta(\omega) = \frac{1}{\pi}\sum_{k=1}^{K} M_k\,&[(\omega+\omega_k)\ln(\omega+\omega_k) + (\omega-\omega_k)\ln|\omega-\omega_k| \\ &-(\omega+\omega_{k-1})\ln(\omega+\omega_{k-1}) - (\omega-\omega_{k-1})\ln|\omega-\omega_{k-1}|]\,. \end{aligned}\tag{4.82}$$

Each segment of finite slope contributes a part to the phase. However, for segments which are remote from ω ($\omega >> \omega_k$ or $\omega << \omega_k$), the contribution is negligible.

The gain-phase relation was first published by Bode in 1945 [8]. Its significance is that the gain and phase of a minimum transfer function are related and cannot be independently specified in circuit design. Given a circuit, we can compute its gain and phase straightforwardly by solving its node equations. However, if the interior of a circuit is unknown, as in the case of modeling of a small-signal electronic device or circuit, we can use Eq. (4.82) to compute the phase from measurements of its gain, provided we have reasons to believe that the transfer function is minimum phase.

4.9 Sensitivity function

In circuit design, it is often desired to find the change in some circuit performance, say, the magnitude or phase, with respect to some change in an element value, say, an inductance or capacitance. By making appropriate changes, it is hoped that a more satisfactory performance is obtained. The sensitivity function will indicate the direction and amount of changes necessary.

Let $F(s)$ be a transfer function. Let x be some element value (real). Let

$$F(j\omega;x) = |F(j\omega;x)|\,\mathrm{e}^{j\theta(\omega;x)}\,.\tag{4.83}$$

We define a ***sensitivity function*** with respect to x as

$$S(F(j\omega);x) = \frac{\dfrac{\mathrm{d}F}{F}}{\dfrac{\mathrm{d}x}{x}}\,.\tag{4.84}$$

It follows that

Lemma 4.1.

$$\Re[S(F(j\omega);x)] = \frac{\frac{\mathrm{d}|F|}{|F|}}{\frac{\mathrm{d}x}{x}} = \textit{gain sensitivity}. \tag{4.85}$$

$$\Im[S(F(j\omega);x)] = x\frac{\mathrm{d}\theta}{\mathrm{d}x} = \textit{phase sensitivity}. \tag{4.86}$$

It appears that to compute the sensitivity function, differentiation of the transfer function with respect to x is required. For large circuits, the explicit form of $F(j\omega;x)$ is usually not available. Numerical differentiation is notoriously inaccurate and unreliable. Fortunately, for linear circuits, including *RLC* and small-signal equivalent circuits, sensitivity can be computed without differentiation. The modified node equations are solved twice, once with an ordinary node matrix and once with its transpose [7, 50, 21].

4.9.1 Computation of sensitivity

Consider an *RLC* circuit of $n+1$ nodes with the $(n+1)$th node assigned to ground. For the time being, assume the circuit consists of resistors, capacitors and inductors only. The case with coupled inductors will be treated separately later. Apply a voltage source of 1V across node 1 and ground. Let the modified node equation be

$$\begin{bmatrix} Y & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} V \\ I_E \end{bmatrix} = \begin{bmatrix} 0 \\ E \end{bmatrix} = \mathbf{e}_{n+1} \ , \tag{4.87}$$

where Y is the $n \times n$ node admittance matrix, V the n-vector of node voltages, I_E the source current, and $\mathbf{e}_{n+1}$ the unit vector with all elements being zeros except the $(n+1)$th, which is unity. For convenience, we will write the modified node equation as

$$Y_m \mathbf{u} = \mathbf{e}_{n+1} \ , \tag{4.88}$$

where $\mathbf{u} = [V\ I_E]^T$ and Y_m is the modified node admittance matrix. Let the kth element of $\mathbf{u}$ be u_k. Let us take the output voltage at node n and the transfer function becomes

$$F(s) = \frac{V_n}{E} = u_n \ , \tag{4.89}$$

which is found by solving Eq. (4.88). Suppose an admittance of interest y is connected between nodes i and j. It appears exactly in four elements of the node admittance matrix Y: as $+y$ in Y_{ii} and Y_{jj}, and $-y$ in Y_{ij} and Y_{ji}. Differentiation of Eq. (4.88) with respect to y yields

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}y} = -Y_m^{-1}\mathbf{e}_{i,j}\mathbf{e}_{i,j}^T\mathbf{u} = -Y_m^{-1}\mathbf{e}_{i,j}(u_i - u_j)\ , \tag{4.90}$$

where $\mathbf{e}_{i,j}$ is a vector of zeros except that the ith element is $+1$ and the jth is -1, and u_i and u_j are the node voltages at nodes i and j. Taking the nth row in Eq. (4.90), we get

$$\frac{\mathrm{d}V_n}{\mathrm{d}y} = -\mathbf{e}_n^T Y_m^{-1}\mathbf{e}_{i,j}(u_i - u_j)\ . \tag{4.91}$$

The product of the first two terms is a row vector. Let it be $\mathbf{q}^T$ and write

$$\mathbf{e}_n^T Y_m^{-1} = \mathbf{q}^T\ . \tag{4.92}$$

Taking transpose, we get

$$[Y_m^{-1}]^T\mathbf{e}_n = \mathbf{q}\ . \tag{4.93}$$

Since the transpose of the inverse of a matrix is the same as the inverse of the transpose, we have a means to find $\mathbf{q}$:

$$Y_m^T\mathbf{q} = \mathbf{e}_n\ , \tag{4.94}$$

which is recognized as a modified node equation in which the modified node admittance matrix is the transpose of that of the original circuit and in which a current source of 1A is applied at node n. We call the circuit whose modified admittance matrix is the transpose of the modified admittance of another circuit the *adjoint circuit* of the second circuit.

Let the components of vector $\mathbf{q}$ be q_k. From Eq. (4.91) and (4.94), we obtain

$$\frac{\mathrm{d}V_n}{\mathrm{d}y} = -(q_i - q_j)(u_i - u_j)\ , \tag{4.95}$$

namely, a product of two voltages, one being across nodes i and j of the original with a 1V voltage source applied at node 1 and the other being across nodes i and node j of the adjoint circuit with a 1A current applied at node n. The unit of q_k is ohm.

Lastly, the sensitivity function with respect to an element admittance y is :

Theorem 4.8.

$$S(V_n;y) = \frac{\frac{\mathrm{d}V_n}{V_n}}{\frac{\mathrm{d}y}{y}} = -\frac{y}{V_n}(q_i - q_j)(u_i - u_j)\ . \tag{4.96}$$

No differentiation is needed. It should be noted that we can follow the same procedure to derive a sensitivity formula for circuits containing transconductances. If g_m is connected between node i and j and is controlled by a voltage across nodes r and s, then it appears as g_m in Y_{ir} and Y_{js} and as $-g_m$ in Y_{is} and Y_{jr}. The unit vectors are $e_{i,j}$ and $e_{r,s}^T$. The rest of the derivation is similar.

Sensitivities with respect to adjustable parameters are needed in the design of circuits by optimization. In almost all optimization methods, the gradient and possibly the Hessian matrix of an objective function are used to determined the direction and magnitude of the next step in the process of finding its minimum in the parameter space.

4.9.2 Computation of group delay

An immediate application of Theorem (4.8) is in the computation of group delay. Let $F(s)$ be a transfer function of an *RLC* circuit with a voltage source applied at node 1 and the output taken at node n. Let

$$F(j\omega) = |F(j\omega)|\, \mathrm{e}^{\theta(\omega)} \,. \tag{4.97}$$

Since the phase function is a function of the element admittances, we can write the group delay $D(\omega)$ as

$$\begin{aligned} D(\omega) &= -\frac{d\theta}{d\omega} = -\sum_k \frac{d\theta}{dy_{C_k}} \frac{dy_{C_k}}{d\omega} - \sum_k \frac{d\theta}{dy_{L_k}} \frac{dy_{L_k}}{d\omega} \\ &= -\frac{1}{\omega}\sum_k C_k \frac{d\theta}{dC_k} - \frac{1}{\omega}\sum_k L_k \frac{d\theta}{dL_k} && (4.98) \\ &= -\frac{1}{\omega}\sum_k \Im\left[S(F(j\omega);C_k)\right] - \frac{1}{\omega}\sum_k \Im\left[S(F(j\omega);L_k)\right] \,, && (4.99) \end{aligned}$$

where y_{C_k} and y_{L_k} are the admittances of the capacitors and inductors and C_k and L_k are the capacitances and inductances, respectively. We note that

$$S(F(j\omega);C_k) = S(F(j\omega);y_{C_k}) \,; \qquad S(F(j\omega);L_k) = -S(F(j\omega);y_{L_k}) \,. \tag{4.100}$$

Using Theorem (4.8), we obtain [50]:

Theorem 4.9. *The group delay of an* RLC *circuit whose transfer function is* $F(j\omega) = V_n/E$ *is given by*

$$\begin{aligned} D(\omega) = &\frac{1}{\omega}\sum_k \Im\left[\frac{j\omega C_k}{V_n}(q_{i_k} - q_{j_k})(u_{i_k} - u_{j_k})\right] \\ &- \frac{1}{\omega}\sum_k \Im\left[\frac{1}{j\omega L_k V_n}(q_{i_k} - q_{j_k})(u_{i_k} - u_{j_k})\right] \,, \end{aligned} \tag{4.101}$$

where the first summation is taken over all capacitances C_k *connected between nodes* i_k *and* j_k*, and the second over all inductances* L_k *connected between nodes* i_k *and* j_k*, and where* q_{i_k} *and* u_{i_k} *are components of* q *and* u *of Eq. (4.94) and Eq. (4.88), respectively, namely*

$$Y_m \mathbf{u} = \mathbf{e}_{n+1} \,, \qquad Y_m^T \mathbf{q} = \mathbf{e}_n \,. \tag{4.102}$$

The theorem says that the group delay consists of two parts: the first part $D_C(\omega)$ is due to the capacitors and the second part $D_L(\omega)$ to the inductors. Secondly, formula (4.101) is valid and applicable to small-signal equivalent circuits as well. Reciprocity does not play any part; linearity does.

Example 4.1. Compute the group delay of the low pass filter of Fig. 4.9 and compare it with that obtained from direct differentiation of the phase. The output is taken at node 3. The node equations of the circuit are:

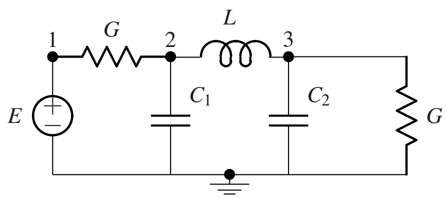

Fig. 4.9 Computation of group delay of a low pass filter

$$\begin{bmatrix} G & -G & 0 & -1 \\ -G & G+j\omega C_1+1/j\omega L & -1/j\omega L & 0 \\ 0 & -1/j\omega L & G+j\omega C_2+1/j\omega L & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ I_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} . \tag{4.103}$$

For each frequency ω, we solve for the node voltages $V_2(j\omega)$ and $V_3(j\omega)$ from the above. Next, for each ω, we solve the node equations of the adjoint circuit below:

$$\begin{bmatrix} G & -G & 0 & 1 \\ -G & G+j\omega C_1+1/j\omega L & -1/j\omega L & 0 \\ 0 & -1/j\omega L & G+j\omega C_2+1/j\omega L & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} . \tag{4.104}$$

for *its* node voltages $q_2(j\omega)$ and $q_3(j\omega)$. The group delay is given by

$$D(\omega) = \frac{1}{\omega}\Im\left[\frac{j\omega C_1}{V_3}q_2V_2 + \frac{j\omega C_2}{V_3}q_3V_3 - \frac{1}{j\omega LV_3}(q_2-q_3)(V_2-V_3)\right] , \tag{4.105}$$

and it is plotted in Fig. 4.10 for $G = 1$ S, $C_1 = C_2 = 1$ F, and $L = 2$ H. The group delay obtained by taking the negative derivative of the phase is:

$$D(\omega) = \frac{2\omega^4+\omega^2+2}{1+\omega^6} , \tag{4.106}$$

shown in dots in Fig. 4.10. The two delay responses match perfectly.

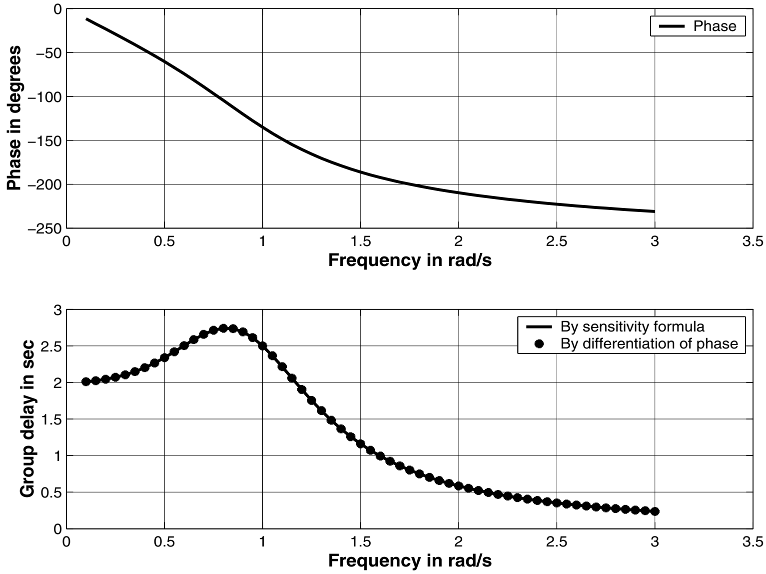

Fig. 4.10 Group delay by direct computation and by differentiation of phase

4.9.2.1 Group delay for circuits with coupled inductors

For the sake of simplicity and clarity, assume there are but two coupled inductors with inductances L_1 and L_2 and a mutual inductance of M. Suppose L_1 is connected across nodes i and j and L_2 across nodes r and s. Then the modified node equation becomes:

$$\begin{bmatrix} Y & e_{i,j} & e_{r,s} & -e_1 \\ -e_{i,j}^T & j\omega L_1 & j\omega M & 0 \\ -e_{r,s}^T & j\omega M & j\omega L_2 & 0 \\ e_1^T & 0 & 0 & 0 \end{bmatrix} \mathbf{u} = \mathbf{e}_{n+3} \,, \tag{4.107}$$

$$Y_m \mathbf{u} = \mathbf{e}_{n+3} \,, \tag{4.108}$$

in which we assume the node admittance Y is $n \times n$ and L_1 is the $\{(n+1),(n+1)\}$ element of Y_m and L_2 the $\{(n+2),(n+2)\}$ element, and M appears in elements $\{(n+1),(n+2)\}$ and $\{(n+2),(n+1)\}$.

Assume we take the output voltage at node n. Let

$$Y_m^T \mathbf{q} = \mathbf{e}_n \tag{4.109}$$

Then the group delay due to the coupled inductors is

$$D_M(\omega) = \frac{1}{\omega}\Im\left[\frac{j\omega L_1}{V_n}q_{n+1}u_{n+1} + \frac{j\omega L_2}{V_n}q_{n+2}u_{n+2} + \frac{j\omega M}{V_n}(q_{n+1}u_{n+2} + q_{n+2}u_{n+1})\right]. \tag{4.110}$$

If there are more than one set of coupled inductors, the delay is the sum of terms similar to the above. The total group delay for the circuit is

$$D(\omega) = D_C(\omega) + D_L(\omega) + D_M(\omega). \tag{4.111}$$

As an example, the phase and group delay of the bandpass filter of Fig. 4.4 are plotted in Fig. 4.11. Note that the group delay is not constant over the passband.

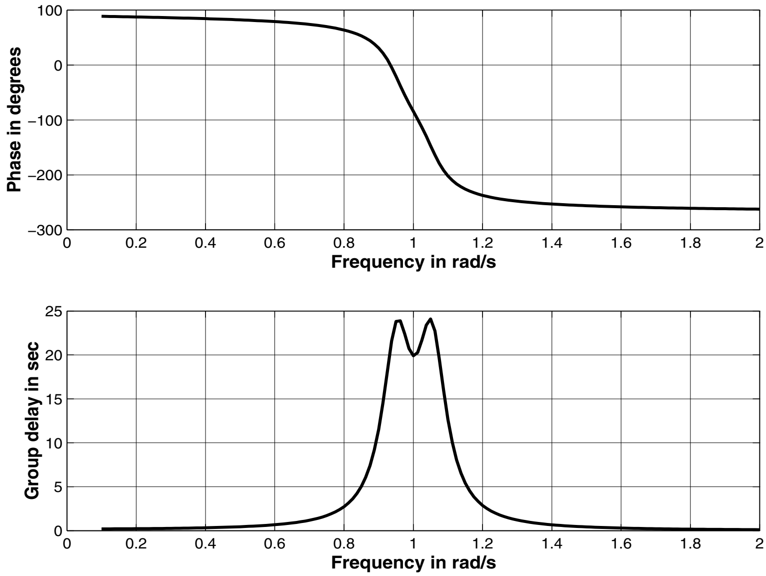

Fig. 4.11 Phase and group delay of a coupled-inductors bandpass filter

4.10 Summary

Because the impulse response of a circuit cannot grow without limit, the poles of circuit functions such as the impedance and transfer function must not have positive real part. These functions are therefore analytic in the right-half s-plane and they have interesting properties along the $j\omega$-axis, namely, in the frequency domain. The real and imaginary parts are related, as are the magnitude and phase. If the real part of a function is specified, for example, the imaginary part, and hence the whole function, is determined to within an odd function. The phase of a minimum phase transfer function can be inferred from the magnitude function plotted on a logarithmic frequency scale, being proportional to the slope of the magnitude in dB. Lastly, the sensitivity of any circuit function with respect to an element value can be computed without taking derivatives. As a consequence, the group delay of a circuit can be found by solving two systems of modified node equations, once with the original node admittance matrix and once with its transpose.

Problems

4.1. Let a circuit be driven by a voltage source and let its input admittance be $Y(s)$. Now replace the voltage source by a current source. Let its input impedance be $Z(s)$. Show that $Y(s) = 1/Z(s)$. See Eq. (4.14).

4.2. In integrated circuit technology, a capacitor consists of two parallel conductors separated by a layer of dielectric. The bottom plate sits atop of an isolation layer which is atop of the substrate. The equivalent circuit of the capacitor is shown in Fig. 4.12 [22]. For a 1 pF capacitor, typical values of the parasitic elements are: $L_s = 0.02\,\text{nH}$, $R_s = 1\,\Omega$, $C_{ox} = 20\,\text{fF}$, $C_{sub} = 20\,\text{fF}$, $R_{sub} = 1000\,\Omega$. Plot the real and imaginary parts of the admittance across the "capacitor" as a function of frequency. Compare the results with those of an ideal capacitor.

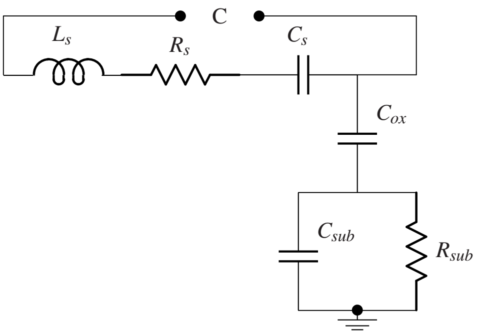

Fig. 4.12 Equivalent circuit of an integrated circuit capacitor at high frequency

4.3. Find an impedance whose real part along the $j\omega$-axis is each of the following:

$$(a)\, R(\omega) = \frac{\omega^4 + 4}{\omega^6 + 7\omega^4 + 31\omega^2 + 25}. \qquad (b)\, R(\omega) = \frac{2\omega^6 + 2\omega^4 + 17\omega^2 + 3}{\omega^6 + \omega^4 + 10\omega^2 + 1}.$$

4.4. Find a minimum phase transfer function $H(s)$ whose magnitude-squared is each of the following:

$$(a)\ |H(j\omega)|^2 = \frac{0.25}{1+\omega^{10}}. \qquad (b)\ |H(j\omega)|^2 = \frac{0.25\omega^{10}}{1+\omega^{10}}.$$

$$(c)\ |H(j\omega)|^2 = \frac{(\omega^2+1)^2}{1+\omega^6}. \qquad (d)\ |H(j\omega)|^2 = \frac{\omega^4+4}{\omega^6+7\omega^4+31\omega^2+25}.$$

4.5. Find the group delay $D(\omega)$ associated with each of the transfer functions of Problem 4.4.

4.6. In the band pass filter of Fig. 4.4, adjust the coupling of the inductors by changing the value of the mutual inductance to see if it is possible to increase the bandwidth of the filter. What is the approximate value of the mutual inductance above which the passband will have ripple? How would you do this experiment in a laboratory?

4.7. Prove Theorem 4.6 and apply it to the impedance function of a capacitor C in parallel with a resistor R. Try the impedance of C, R and an inductance L in parallel.

4.8. Prove the Hilbert transform pair of (4.71).

4.9. Derive Eq. (4.74). Suggestion: Integrate Eq. (4.70) by parts to get:

$$X(\omega) = \frac{1}{\pi}\left\{ R(\Omega)\ln(\Omega-\omega)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \ln(\Omega-\omega)\frac{\mathrm{d}R}{\mathrm{d}\Omega}\,\mathrm{d}\Omega \right\}. \tag{4.112}$$

There is a logarithmic singularity at $\Omega = \omega$. Break up the integration into two parts: one from $\Omega = -\infty$ to $\omega - \delta$, and one from $\Omega = \omega + \delta$ to ∞. Take the limit as $\delta \to 0$ and get

$$X(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \ln|\Omega-\omega|\,\frac{\mathrm{d}R}{\mathrm{d}\Omega}\,\mathrm{d}\Omega\,, \tag{4.113}$$

which will lead to Eq. (4.74) if we express $\ln|\Omega-\omega|$ as a sum of its even and odd parts.

4.10. Let $H(s)$ be a minimum phase transfer function. Let $\ln H(j\omega) = \alpha(\omega) + j\beta(\omega)$. Show that

$$\int_0^{\infty} \frac{\beta(\omega)}{\omega}\,d\omega = \frac{\pi}{2}(\alpha(\infty) - \alpha(0)) \tag{4.114}$$

Offer an interpretation of this result in terms of gain and phase.

4.11. Derive an expression of group delay for a circuit containing coupled inductors, as given in Eq. (4.110).

4.12. Let $Z(s)$ be the input impedance of an RLC circuit. Let L be the inductance of an inductor connected between two nodes other than the input nodes. Show that

$$\frac{dZ}{dL} = j\omega\frac{I_L^2}{I_{in}^2} \tag{4.115}$$

where I_L is the current in the inductor and I_{in} is the input current.

4.13. Compute the frequency response $H(j\omega) = V_o/E$ of the fifth order Butterworth low-pass filter of Fig. 4.13. Plot the gain in dB, phase in radian and group delay in second over a log frequency scale: $p = \log\omega$, in the range $p = [-1, 1]$. We will learn how to design such a filter in Chapter 10. Can you infer the phase from the gain?

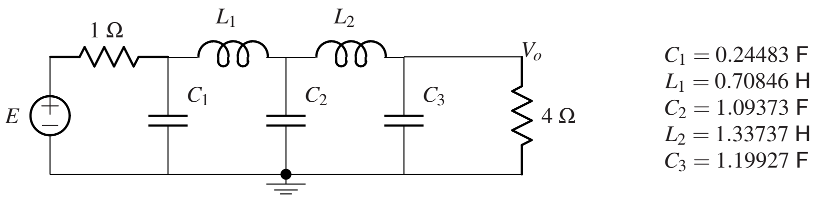

Fig. 4.13 A Butterworth lowpass filter of order 5

4.14. Compute the frequency response $H(j\omega) = V_o/E$ of the fifth order Chebyshev low-pass filter of Fig. 4.14. Plot the gain in dB , phase in radian and group delay in second over a log frequency scale: $p = \log\omega$, in the range $p = [-1, 1]$. Separately plot an expanded view of the passband to show the equal-ripple characteristics.

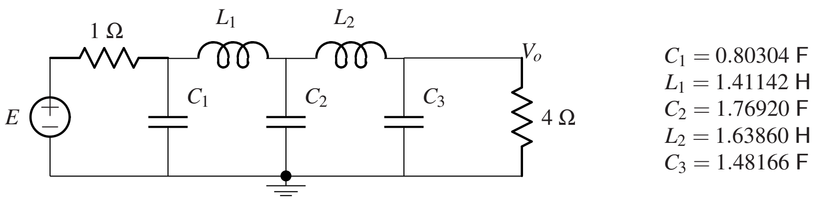

Fig. 4.14 A Chebyshev lowpass filter of order 5

4.15. Repeat Prob. 4.4.14 for the Cauer or elliptic low-pass filter shown in Fig. 4.15. The element values are:

$C_1 = 0.66300$ F, $C_1' = 0.73948$ F, $L_1 = 0.75056$ H,
$C_2 = 1.49161$ F, $C_2' = 0.23437$ F, $L_2 = 1.13735$ H, $C_3 = 0.96858$ F.

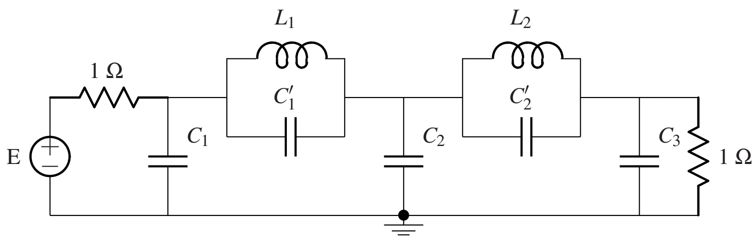

Fig. 4.15 A fifth order Cauer or elliptic low-pass filter

4.16. Compare the frequency responses of the Butterworth, Chebyshev and Cauer low-pass filters. If filtering is important, which of the three filters would you use? If constant group delay is important, which one would you use?

4.17. Compare the pulse response of the Butterworth filter with that of the Chebyshev filter of Problems 4.13 and 4.14, respectively. The input $e(t) = 0.5(1 + \cos((4t/5 - 0.5)\pi))$ for $0 \le t < 5/4$; $e(t) = 1$ for $5/4 \le t < 15/4$; $e(t) = 0.5(1 + \cos((4t/5 - 0.5)\pi))$ for $15/4 \le t < 5$; and $e(t) = 0$ for $t \ge 5$. Note the delay and dispersion of the pulse in each. Use Backward Euler integration of the modified node equations.

4.18. The circuit shown in Fig. 4.16 is known as a constant-R lattice in which $Z_A(s)Z_B(s) = R^2$. (a) Show that its input impedance between $1-2$ is R. (b) Show that its transfer function is

$$F(s) = \frac{V_{out}}{E} = \frac{1}{2}\frac{R - Z_A(s)}{R + Z_A(s)}. \tag{4.116}$$

Let $Z_A(s)$ be a parallel LC circuit. (1) Show the transfer function is all-pass. (2) Obtain the expressions for the phase and group delay. (3) Let L=1 H C=1 F and R=1 Ω . Plot the gain, phase and group delay as a function of frequency. (4) Connect n constant-R sections in cascade. Show that the overall transfer function is the product of the n transfer functions. Show also that the input impedance remains unchanged, namely, it is $R\,\Omega$, independent of n.

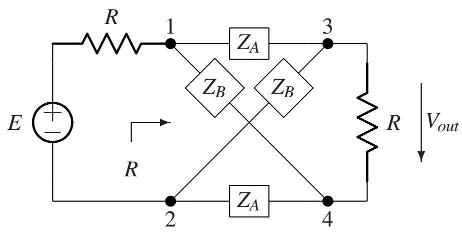

Fig. 4.16 A constant-R lattice. The input impedance is R as long as $Z_A(s)Z_B(s) = R^2$

4.19. An ideal lossless transmission line has a transfer function $F(s) = 0.5e^{-s\tau}$, where τ is the delay. It is seen that the phase is a linear function of ω and the group delay is constant. Consider a line of 1 meter long with $L = 1$H/M and $C = 1$F/M. Approximate the line with n sections of LC ladder with L in the series arm and C in shunt. Terminate the circuit in a one-ohm resistor at both ends. Compute the magnitude, phase and group delay of the circuit and compare with the ideal results. The results for the case of $n = 5$ are shown in Fig. 4.17. It is seen that the magnitude and delay are correctly modeled at low frequencies but the approximation is poor at high frequencies. Try $n = 10$ and see if the results are any better.

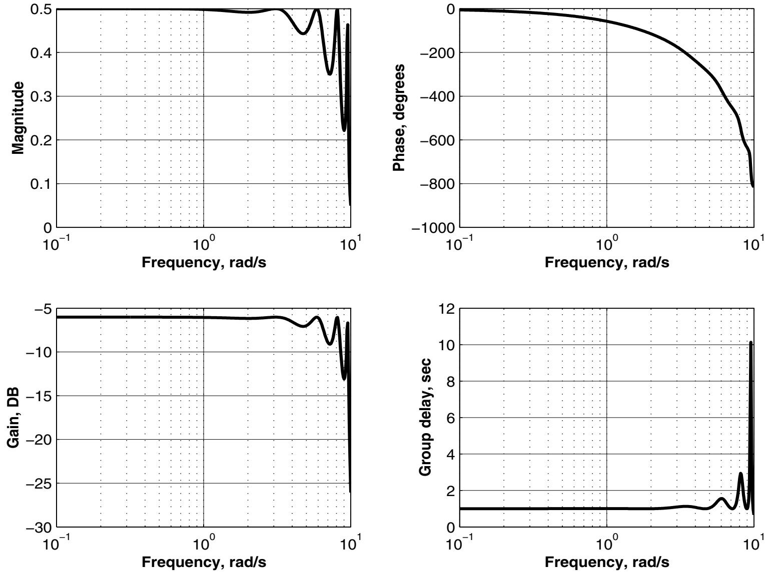

Fig. 4.17 Frequency response of a five-section *LC* ladder to model a transmission line

4.20. Another circuit whose transfer function approximates that of a delay line of Problem 4.19 is the constant-R lattice of Problem 4.18. Its transfer function is:

$$F(s) = \frac{1}{2}\frac{s^2-6s+12}{s^2+6s+12}\,. \tag{4.117}$$

Compute and plot the gain and group delay. Find a circuit to realize $F(s)$. Cascade five sections of such constant-R lattices and compare the results with those of Problem 4.19. Discuss the relative merits of the two designs.

4.21. In integrated circuit technology, a length of connection, called an interconnect, is often modeled as a segment of *RC* transmission line, which in turn is approximated as a number of *RC* ladders connected in cascade. Each ladder consists of a series resistor and a shunt capacitor. Let the resistance of each resistor be 1 Ω and let the capacitance of each capacitor be 1 F. (a) Using symbolic mathematics, find the transfer function of a model with five ladder sections. The input is a voltage source and the output is taken across the last capacitor. (b) Obtain the frequency response of the circuit by solving the node equations in the frequency domain. Compare it with that obtained by evaluating the transfer function for a range of frequencies. (c) Can you infer the phase response from the gain response? (d) Compute the group delay response without taking derivatives and compare it with that obtained by taking derivative of the phase function. (e) Find the poles and zeros of the transfer function. Are you surprised that the poles are all negative real? See Chapter 6.

4.22. The *RC* ladder model of the integrated circuit interconnect of Problem 4.21 can also be used to represent a lossy telegraphy cable of the olden days. As you see from the frequency response in Fig. 4.18, the output drops rapidly to zero as frequency increases. To extend the useful frequency range, or to improve the frequency response, Oliver Heaviside, I. Pupin, and George Campbell, in the 1880s, proposed to add a "loading coil" periodically along the cable. Let us repeat their experiment by adding an inductor of inductance 6 H in series with each resistor in the five section ladder. Compute the frequency response. You will see an improved response. The mathematical theory was formally derived by Bode in the 1930s [8] (see Sect. 8.3). Another way of looking at the results is that the length of cable can be extended to provide acceptable telegraphy service to a more distant city by adding the loading coils. How much farther a city can be reached by the addition of the loading coils if each ladder section represents a unit length? We define acceptable service as the situation where the output voltage at the end of the cable at frequency 0.1 rad/s drops to 50% of the input. (Answer: About 40% farther.)

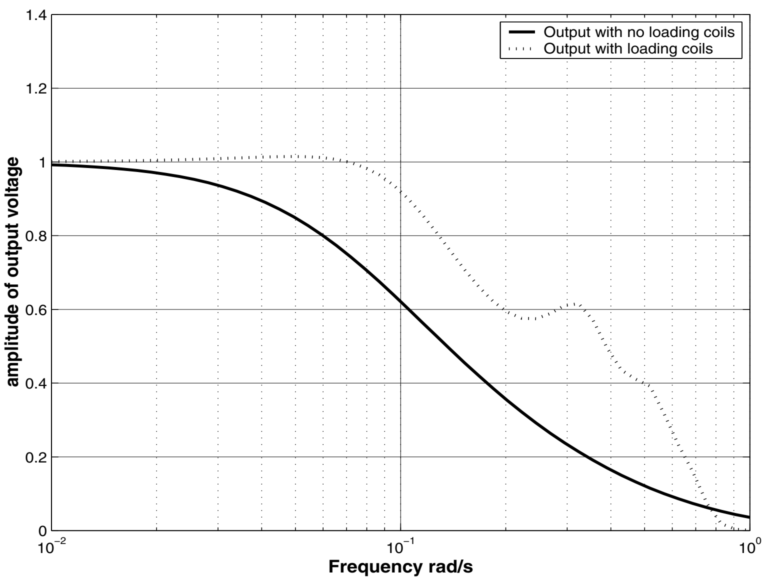

Fig. 4.18 Frequency response of an *RC* interconnect or cable with and without loading coils

4.23. In the design of active filters, it is important to determine the sensitivity of the transfer function with respect to variations of the active elements, which are usually represented by transconductances. Derive an expression for the sensitivity of the transfer function with respect to a transconductance g_m connected between nodes i and j and controlled by a voltage between nodes r and s.

Chapter 5
The Impedance Function

We saw in the last chapter that the impedance function of a circuit is a rational function in s with special properties: e.g., its poles and zeros cannot be in the right half of the s-plane. This condition is necessary but not sufficient. For example, $F(s) = (s+1)^4/(s+2)^4$ is an impedance function but $F(s) = (s+1)^5/(s+2)^5$ is not. In this chapter we will derive the necessary and sufficient conditions for a rational function to be an impedance or admittance function of an *RLC* circuit including coupled inductors.

5.1 Preliminaries

We recall from Sect. 5.1 that the inductance matrix of two coupled inductors is positive semi-definite. If

$$L_m = \begin{bmatrix} L_a & M \\ M & L_b \end{bmatrix} \tag{5.1}$$

is the inductance matrix, then

$$x^T L_m x \geq 0 \tag{5.2}$$

for all real-valued vectors x. Recall also that L_m is symmetric.

Theorem 5.1. *Let L_m be a real, symmetric matrix which is positive semi-definite with respect to a real-valued vector. Then L_m is positive semi-definite with respect to a complex vector and its conjugate, namely*

$$X^T L_m X^* \geq 0 \tag{5.3}$$

for any $X = X_r + jX_i$.

Proof. Direct substitution yields

$$[X_r^T + jX_i^T]L_m[X_r - jX_i] = X_r^T L_m X_r + X_i^T L_m X_i + j(X_i^T L_m X_r - X_r^T L_m X_i)$$
$$= X_r^T L_m X_r + X_i^T L_m X_i \geq 0\,, \tag{5.4}$$

since $X_i^T L_m X_r = X_r^T L_m X_i$ and each of the quadratic forms in the last expression is positive semi-definite. □

5.2 Positive real function

Consider a circuit consisting of resistors, inductors, coupled inductors and capacitors. Let R_k be the resistances of the resistors, L_k the inductances of the inductors, L_{m_k} be the inductance matrices of the coupled inductors, and C_k the capacitances of the capacitors. Let I_{R_k} and V_{R_k} be the current and voltage of R_k, and similarly for I_{L_k}, V_{L_k}, I_{m_k}, V_{m_k}, I_{C_k} and V_{C_k}. Let a current source J be connected across node 1 and ground. Denote its voltage as V_1 and the impedance it sees as $Z(s)$, where $s = \sigma + j\omega$ is the complex frequency. Since the element currents and their complex conjugates satisfy *KCL* and the element voltages satisfy *KVL*, by Tellegan's theorem, we have

$$J^* V_1 = \sum_k I_{R_k}^* V_{R_k} + \sum_k I_{L_k}^* V_{L_k} + \sum_k [I_{m_k}^*]^T V_{m_k} + \sum_k I_{C_k}^* V_{C_k}\,. \tag{5.5}$$

After substitution of $V_{R_K} = R_k I_{R_k}$, etc., we have

$$Z(s) = \frac{1}{JJ^*}\left\{\sum_k R_k I_{R_k} I_{R_k}^* + s\sum_k L_k I_{L_k} I_{L_k}^* \right.$$
$$\left. + s\sum_k [I_{m_k}^*]^T L_{m_k} I_{m_k} + \frac{1}{s}\sum_k \frac{1}{C_k} I_{C_k} I_{C_k}^* \right\}, \tag{5.6}$$

and we have related the driving point impedance function to the energy of the circuit. It is important to note that the voltages and currents are all functions of $s = \sigma + j\omega$. The third sum is non-negative by Theorem 5.1, and the other three sums are each non-negative, for all values of σ and ω.

In a dual manner, if the circuit is driven by a voltage source E, then the driving point admittance seen by E is:

$$Y(s) = \frac{1}{EE^*}\left\{\sum_k G_k V_{R_k} V_{R_k}^* + s\sum_k C_k V_{C_k} V_{C_k}^* \right.$$
$$\left. + \frac{1}{s}\sum_k \frac{1}{L_k} V_{L_k} V_{L_k}^* + s\sum_k [I_{m_k}^*]^T L_{m_k} I_{m_k} \right\}, \tag{5.7}$$

where $G_k = 1/R_k$. It has the same general form as the impedance $Z(s)$. It follows they have the same general functional properties.

In Eq. (5.6), when s is real, $Z(s)$ is real. Now take the real part on both sides of Eq. (5.6) and we get:

$$\Re Z(s) = \frac{1}{JJ^*}\left\{\sum_k R_k I_{R_k} I_{R_k}^* + \sigma \sum_k L_k I_{L_k} I_{L_k}^* + \sigma \sum_k [I_{m_k}^*]^T L_{m_k} I_{m_k} + \frac{\sigma}{\sigma^2+\omega^2} \sum_k \frac{1}{C_k} I_{C_k} I_{C_k}^* \right\}. \tag{5.8}$$

An important observation is that $\Re Z(s) \geq 0$ whenever $\sigma \geq 0$. We now define a *positive real function*:

Definition 5.1. A rational function $F(s)$ is called a positive real function if and only if[1]

1. $F(s)$ is real when s is real; and
2. $\Re F(s) \geq 0$ when $\Re s \geq 0$.

We state one of the most fundamental theorems in circuit theory:

Theorem 5.2. *The driving point impedance (admittance) of an* RLC *circuit is a positive real function.*

This theorem plays a central role in the synthesis of filters and frequency selective circuits. It holds for all circuits consisting of resistors, inductors, coupled inductors and capacitors.

As we have noted earlier, the impedance and admittance functions of a circuit have similar properties. Engineers have coined the term *immittance* to refer to either.

5.2.1 Small-signal active circuits

If a circuit contains transconductances, the input impedance in general is not a positive real function. As we saw in Sect. 2.9 it is possible to make up a circuit with transconductances such that the input impedance has a negative real part. More formally, let an element be a transconductance g_m whose terminal voltage is $V_1 = V_{1r} + jV_{1i}$ and whose current is controlled by $V_2 = V_{2r} + jV_{2i}$. When we apply Tellegen's theorem to the circuit, we have a term $g_m V_1^* V_2$ among the sums of terms associated with other elements. When the real part is taken, this term becomes $g_m(V_{1r}V_{2r} + V_{1i}V_{2i})$. Its sign is undetermined. It follows that the input impedance may or may not be a positive real function.

[1] The definition can be extended to include irrational functions if item one is changed to read "$F(s)$ is real when s is positive real." See Problem 5.9.

5.3 Properties of positive real function

By definition, a positive real function $F(s)$ maps the right-half of the s plane to the right-half of the $F(s)$ plane. As a consequence, we have

Theorem 5.3. *Let $Z(s) = |Z(s)|\,e^{j\theta}$ be an impedance function. Let $s = |s|\,e^{j\phi}$. Then*

$$|\theta| \le \frac{\pi}{2} \quad \textit{when} \quad |\phi| \le \frac{\pi}{2}. \tag{5.9}$$

So the phase angle of an impedance can never be greater than $\pi/2$ in absolute value. Also, $F(s) = s$ is a positive real function but $F(s) = s^2$ is not. $F(s) = \sqrt{s}$ maps the right-half s-plane to the right-half F-plane, but it is not real when $s = -1$. So it is not a positive real function in a strict sense. There are other obvious results, as follows.

Theorem 5.4. *The sum of two positive real functions is a positive real function.*

Theorem 5.5. *The reciprocal of a positive real function is a positive real function.*

Theorem 5.6. *A positive real function of a positive real function is a positive real function.*

Theorem 5.7. *A positive real function cannot have any poles in the right-half s plane.*

Proof. Let $F(s)$ be a positive real function and assume it has a pole at $p = p_r + jp_i$ such that $p_r > 0$. In the neighborhood of p, $F(s)$ can be approximated by $r/(s-p)$ where $r = r_r + jr_i$ is the residue. Taking real parts, $\Re F(s) \approx [r_r(\sigma - p_r) + r_i(\omega - p_i)]/\,|s-p|^2$. There exist values of σ and ω such that $\Re F(s) < 0$, contradicting that $F(s)$ is a positive real function. Therefore $F(s)$ cannot have any poles in the right-half s plane. □

Theorem 5.8. *A positive real function cannot have any zeros in the right-half s plane.*

Theorem 5.9. *A positive real function may have poles on the $j\omega$-axis, but each must be simple and the residue at each must be positive real.*

Proof. Let $F(s)$ be a positive real function with a pole at $s = j\omega_o$ of order n. In the neighborhood of the pole, $F(s)$ can be approximated by $F(s) \approx r/(s - j\omega_o)^n$. Along the circumference of a semicircle in the right-half s plane, $s = j\omega_o + \rho e^{j\phi}$ with $-\pi/2 \le \phi \le \pi/2$, and with $\rho \approx 0$. Near the pole, $F(s) \approx r/\rho^n e^{-jn\phi}$. Since $F(s)$ is a positive real function by hypothesis, as ϕ varies from $-\pi/2$ to $\pi/2$, the real part of $F(s)$ must be non-negative. We conclude that r must be positive real and $n = 1$. □

Theorem 5.10. *Let $F(s) = P(s)/Q(s)$ be a rational positive real function. Then the degree of $P(s)$ and the degree of $Q(s)$ cannot differ by more than one.*

Theorem 5.11. *Let $F(s)$ be a positive real function. Let $F(j\omega) = R(\omega) + jX(\omega)$. Then $R(\omega) \geq 0$ for all ω.*

So the real part of an impedance function along the $j\omega$-axis is non-negative.

It appears that the testing of a function $F(s)$ to see if it is a positive real function requires that we check its real part for all values of s in the right-half s plane. This is a substantial effort. Fortunately, by making use of properties of analytic functions of a complex variable, we reduce this effort to checking the real part along the $j\omega$-axis. The mathematical basis for this is given next.

5.4 Necessary and sufficient conditions

We need a theorem from the theory of functions of a complex variable [28].

Theorem 5.12. *Let $F(s)$ be analytic in a region R of the s plane. Let its boundary be C. Then the minimum real part of $F(s)$ for all s in R is on C.*

It will be used in the proof of the following important theorem in circuit theory [29, 68].

Theorem 5.13. *The necessary and sufficient conditions for a function $F(s)$ to be a positive real function are:*

1. *$F(s)$ is real when s is real.*
2. *$F(s)$ is analytic in the right-half of the s plane.*
3. *Let $F(j\omega) = R(\omega) + jX(\omega)$. $R(\omega)$ is non-negative for all ω.*
4. *Poles of $F(s)$ on the $j\omega$-axis are simple with positive real residues.*

Proof. The necessity part has already been proved by definition and Theorems 5.7, 5.11, and 5.9. As to the sufficiency part, let $F(s)$ be a function that satisfies the four conditions. Condition (1) is part of the definition. Condition (2) implies that Theorem 5.12 applies and the minimum real part of $F(s)$ occurs on the $j\omega$-axis, which is the boundary of the right-half s plane. Condition (3) implies that $\Re F(s) \geq 0$ in the right-half s plane. If $F(s)$ has a pole on the $j\omega$-axis, condition (4) ensures that the real part of $F(s)$ is non-negative in its neighborhood that is in the right-half s plane. So we have that $F(s)$ is real when s is real and $\Re F(s) \geq 0$ everywhere in the right-half s plane, including the boundary, By definition, $F(s)$ is a positive real function. □

Example 5.1. The following functions are positive real or not for reason indicated. We use the abbreviation ”p.r.” to mean positive real.

$$F(s) = \frac{s+1}{s+2}. \qquad \text{(p.r.)}$$

$$F(s) = \frac{(s+1)^4}{(s+2)^4}. \qquad \text{(p.r.)}$$

$$F(s) = \frac{(s+1)^5}{(s+2)^5}. \qquad \text{(Not p.r. } \theta > \pi/2 \text{ for some } \omega.)$$

$$F(s) = \frac{s^2+s+1}{s^2+2s+4}. \qquad \text{(Not p.r. } \Re F(j\omega) < 0 \text{ for some } \omega.)$$

$$F(s) = \frac{4s^3+s^2+3s+2}{(s^2+1)(s+1)}. \qquad \text{(Not p.r. Residue at } s = j \text{ not positive real.)}$$

5.5 Useful theorems

The following three theorems are important in the synthesis of transfer functions [29, 68].

Theorem 5.14. *Let $F(s) = (m_1+n_1)/(m_2+n_2)$ be a positive real function, where m_1 is the even part of the numerator polynomial of $F(s)$ and n_1 is its odd part. Similarly, m_2 and n_2 are the even and odd parts, respectively, of the denominator polynomial. Then*

$$G(s) = \frac{m_1+n_2}{m_2+n_1} \tag{5.10}$$

is a positive real function.

The proof is left as an exercise in Problem 5.1. Using the same notations, we note that the real part of a positive real function $F(s) = (m_1+n_1)/(m_2+n_2)$ along the $j\omega$ axis is obtained from:

$$\Re Z(j\omega) = \left.\frac{m_1m_2 - n_1n_2}{m_2^2 - n_2^2}\right|_{s=j\omega} \tag{5.11}$$

The even polynomial

$$E(s^2) = m_1m_2 - n_1n_2 \tag{5.12}$$

is called the *ensignant*[68] of $F(s)$ and since $F(s)$ is positive real,

$$E(-\omega^2) = m_1m_2 - n_1n_2|_{s^2=-\omega^2} \geq 0 \tag{5.13}$$

for all ω. $E(s^2)$ plays an important role in the synthesis of transfer functions as will be described in Chapter 9.

Theorem 5.15. *If $E(s^2)$ is the ensignant of a positive real function, then its zeros have the following properties:*

1. *Real zeros occur in symmetric pairs with respect to the $j\omega$ axis;*

2. Complex zeros occur in quadrantal symmetry; and
3. Zeros on the $j\omega$ axis occur in even order.

Proof. Properties 1 and 2 follow from the fact that $E(s^2)$ is an even polynomial in s. Each symmetrical pair of real zeros provides a factor $(a^2 - s^2)$ where a is a real constant. At $s = j\omega$, this factor becomes $(a^2 + \omega^2) \geq 0$ for all ω. Each set of quadrantal zeros provides a factor $(s^2 + as + b)(s^2 - as + b)$ where a and b are positive. At $s = j\omega$, this factor becomes $(b - \omega^2)^2 + a^2\omega^2 \geq 0$ for all ω. Lastly, let $j\omega_o$ be a zero on the $j\omega$ axis of order n. Then $E(s^2)$ has a factor $(s^2 + \omega_o^2)^n$. There can be no other zeros. So $E(-\omega^2)$ must have the following form:

$$E(-\omega^2) = \prod_k (a_k^2 + \omega^2)((b_k - \omega^2)^2 + a_k^2\omega^2)(-\omega^2 + \omega_k^2)^n \tag{5.14}$$

By hypothesis, $E(s^2)$ is the ensignant of a positive real function. $E(-\omega^2) \geq 0$ and it follows that n must be even. □

The following theorem is important to the synthesis of transfer functions and in the terminal characterization of a two-port.

Theorem 5.16. *Let*

$$\rho(s) = \frac{Z(s) - 1}{Z(s) + 1}. \tag{5.15}$$

Then the necessary and sufficient conditions that $Z(s)$ is p.r. are

1. $\rho(s)$ is real when s is real;
2. $\rho(s)$ is analytic in the right-half s-plane; and
3. $|\rho(j\omega)| \leq 1$ for all ω.

Proof. (”If” part:) If $Z(s)$ is p.r., then $Z(s)$ is real when s is real. When $Z(s)$ is real, $\rho(s)$ is real and (1) follows. Since $Z(s)$ is p.r., $Z(s) + 1$ is p.r. and does not have any zeros in the right-half s-plane. Therefore (2) follows. Next, let $Z(j\omega) = R(\omega) + jX(\omega)$. Then

$$|\rho(j\omega)|^2 = \frac{(R(\omega) - 1)^2 + X^2(\omega)}{(R(\omega) + 1)^2 + X^2(\omega)}. \tag{5.16}$$

Since $R(\omega) \geq 0$, condition (3) follows.
(”Only if” part:) Write

$$Z(s) = \frac{1 + \rho(s)}{1 - \rho(s)}. \tag{5.17}$$

If $\rho(s)$ is real when s is real, then $Z(s)$ is real when s is real, and $Z(s)$ satisfies part 1 of the definition of a p.r. function. Condition (2) implies that the maximum modulus of $\rho(s)$ occurs on the $j\omega$ axis. Condition (3) implies that $|\rho(s)| \leq 1$ for all s in the right-half s-plane. Let $\rho(s) = r(\sigma, \omega) + jx(\sigma, \omega)$. Now

$$Re[Z(s)] = \frac{1 - (r^2(\sigma, \omega) + x^2(\sigma, \omega))}{(1 - r(\sigma, \omega))^2 + x^2(\sigma, \omega)}. \tag{5.18}$$

Since $r^2(\sigma,\omega)+x^2(\sigma,\omega) = |\rho(s)|^2 \leq 1$ for $Re[s] \geq 0$, $Re[Z(s)] \geq 0$ when $Re[s] \geq 0$. So $Z(s)$ is positive real. □

5.6 Impedance removal

In the next few chapters, we will be concerned with synthesis of impedance (admittance) functions. Given a positive real function, we wish to find a circuit whose impedance is that given function. Our strategy is to remove a part of the given function as partial impedance, leaving a remainder which is of lower order. This process is continued until we have a remainder that is the impedance of a single element. The following theorems are self-evident and are useful for this purpose.

Theorem 5.17. *Let $F(s)$ be a positive real function with a pole at $s = \infty$. Let the residue at that pole be k_∞. Then*

$$F'(s) = F(s) - k_\infty s \tag{5.19}$$

is a positive real function with no pole at $s = \infty$.

Theorem 5.18. *Let $F(s)$ be a positive real function with a pole at $s = 0$. Let the residue be k_0. Then*

$$F'(s) = F(s) - \frac{k_0}{s} \tag{5.20}$$

is a positive real function with no pole at $s = 0$.

Theorem 5.19. *Let $F(s)$ be a positive real function with a pair of poles on the $j\omega$-axis at $s = \pm j\omega_o$. Then*

$$F'(s) = F(s) - \frac{2ks}{s^2+\omega_o^2} \tag{5.21}$$

where k is the residue, is a positive real function.

Theorem 5.20. *Let $F(s)$ be a positive real function such that its minimum real part along the $j\omega$-axis is R_{min} at ω_o. Then*

$$F'(s) = F(s) - R_{min} \tag{5.22}$$

is a positive real function and $F'(j\omega_o) = jX(\omega_o)$.

5.7 Remarks

In this chapter, we showed that the impedance function of every *RLC* circuit is a positive real function. Does every positive real rational function correspond to an impedance of some circuit? The answer is affirmative as we will show in the following chapters. So the condition that an impedance function of an *RLC* circuit is a positive real function is both necessary and sufficient.

Problems

5.1. Let $F(s) = (m_1 + n_1)/(m_2 + n_2)$ be a positive real function, where m_1 is the even part of the numerator polynomial of $F(s)$ and n_1 is its odd part. Similarly, m_2 and n_2 are the even and odd parts, respectively, of the denominator polynomial. Show that $G(s) = (m_1 + n_2)/(m_2 + n_1)$ is a positive real function.

5.2. Show that the following functions are all positive real functions.

$$(a)\ F(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)}. \qquad (b)\ F(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}.$$

$$(c)\ F(s) = \frac{s^2+2s+2}{s^3+3s^2+5s+3}. \qquad (d)\ F(s) = \frac{s^3+s^2+2s+1}{s^2+s+1}.$$

5.3. Are any of the following functions positive real functions?

$$(a)\ F(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)}. \qquad (b)\ F(s) = \frac{(s^2+2)(s^2+3)}{s(s^2+1)(s^2+4)}.$$

$$(c)\ F(s) = \frac{s^2+s+2}{s^3+2s^2+2s+1}. \qquad (d)\ F(s) = e^s.$$

5.4. For what values of a will $F(s) = [(s+a)/(s+1)]^{100}$ be a positive real function?

5.5. For what integer values of n will $F(s) = [(s+1)/(s+1.01)]^n$ be a positive real function?

5.6. For what values of a will $F(s) = (s^2+s+a)/(s^3+2s^2+2s+1)$ be a positive real function?

5.7. for what values of c will the following functions be positive real?

$$(a)\ F(s) = \frac{s^2+2s+c}{s^2+s+1}. \qquad (b)\ F(s) = \frac{s^2+s+1}{s^2+2s+c}.$$

5.8. Imagine yourself to be a teaching assistant and you have been asked to provide examples of positive real functions, each with a stated property as follows.

1. A positive real function whose ensignant $E(s^2)$ has two zeros at $\pm j$.
2. A positive real function whose ensignant $E(s^2)$ has two zeros at ± 1.
3. A positive real function whose ensignant $E(s^2)$ has four zeroes at $\pm 1 \pm j$.

Please oblige. Obviously, there are no unique answers. Find a simplest positive real function in each case.

5.9. The definition of a positive real function can be extended to include irrational functions such as $\sqrt{s}$, as follows.

Definition 5.2. A function $F(s)$ of a complex variable s is a positive real function if and only if

1. $F(s)$ is real when s is positive real; and
2. The real part of $F(s)$ is non-negative when the real part of s is non-negative.

Under the extended definition, show that if $Z(s)$ is a positive real function, then $F(s) = \sqrt{Z(s)}$ is a positive real function. Second, show that if $Z_a(s)$ and $Z_b(s)$ are positive real functions, then $F(s) = \sqrt{Z_a(s)Z_b(s)}$ is a positive real function.

5.10. Under the extended definition of a positive real function, which of the following functions are positive real functions? All constants are positive real.

$$(a)\ F(s) = \sqrt{R/sC}. \qquad (b)\ F(s) = \sqrt{1+\sqrt{s}+s}.$$

$$(c)\ F(s) = \sqrt{\frac{sL+R}{sC+G}}. \qquad (d)\ F(s) = \sqrt{\frac{1+\sqrt{s}}{1+2\sqrt{s}}}.$$

$$(e)\ F(s) = \sqrt{s^2+s+1}. \qquad (f)\ F(s) = \sqrt{\frac{s^2+s+1}{s^2+2s+2}}.$$

5.11. Consider an infinite RC ladder whose series arm is a resistor of resistance R and whose shunt arm is a capacitor of capacitance C. Show that the input impedance of the ladder is

$$Z(s) = \frac{R}{2}\left(1+\sqrt{1+\frac{4}{sRC}}\right). \tag{5.23}$$

Show that $Z(s)$ is a positive real function under the extended definition.

5.12. Consider an infinite LC ladder whose series arm is an inductor of inductance L and whose shunt arm is a capacitor of capacitance C. Show that the input impedance of the ladder is

$$Z(s) = \frac{sL}{2}\left(1+\sqrt{1+\frac{4}{s^2LC}}\right). \tag{5.24}$$

Show that $Z(s)$ is a positive real function under the extended definition. Plot the real part of $Z(j\omega)$ as a function of ω.

Chapter 6
Synthesis of Two-Element-Kind Impedances

Beginning with this chapter, we shall demonstrate that given a positive real function, there exists an *RLC* circuit whose impedance is that given function. In other words, we want to show that the converse of the statement that an impedance function is a positive real function is also true. The systematic procedure by which the circuit is found is called a synthesis method. For each class of impedance functions, there are many synthesis methods and each will lead to a different realization. The realized circuits all have the same impedance, but some configurations are preferred because they have better spread of element values or fewer ungrounded nodes.

Interestingly, if we start with an arbitrary circuit and obtain its impedance, there is no guarantee that any of the synthesis methods will recover that particular circuit. In this sense, the set of known synthesis methods is not complete.

We begin with the synthesis of inductor-capacitor (*LC*) impedance, followed by resistor-capacitor (*RC*) and resistor-inductor (*RL*) impedances. In the next chapter, we will take up synthesis of *RLC* impedance. As before, we will concentrate on impedance. The synthesis of admittance function is largely identical.

6.1 *LC* impedance function

In the last chapter, we derived an expression for the impedance of an *RLC* circuit and related it to the energy of the circuit (Eq. (5.6)). In the case of an *LC* circuit, the term associated with the resistors is absent and the expression becomes:[1]

$$Z(s) = \frac{1}{JJ^*}\left\{ s\sum_k L_k I_{L_k} I_{L_k}^* + \frac{1}{s}\sum_k \frac{1}{C_k} I_{C_k} I_{C_k}^* \right\}. \tag{6.1}$$

[1] We will omit coupled inductors for simplicity. As we saw in the last chapter, they can be treated in the same way as inductors.

In addition to being a positive real function, the impedance function $Z(s)$ has other analytic properties which we will deduce from this expression.

6.1.1 Necessary and sufficient conditions - Foster realizations

From Eq. (6.1), we note firstly that $Z(s)$ is an odd function of s. This implies that $Z(s)$ must be an even polynomial over an odd polynomial, or vice versa, and it can be written as

$$Z(s) = \frac{sP(s^2)}{Q(s^2)} \quad \text{or} \quad \frac{P(s^2)}{sQ(s^2)} . \tag{6.2}$$

An even polynomial has its roots symmetrically placed with respect to the $j\omega$-axis. However, a positive real function cannot have any poles in the right-half s-plane. We conclude therefore:

Lemma 6.1. *The poles of an* LC *impedance function must all be on the $j\omega$-axis and are all simple with positive real residues.*

In a dual manner, an *LC* admittance function has the same property:

Lemma 6.2. *The poles of an* LC *admittance function must all be on the $j\omega$-axis and are all simple with positive real residues.*

As a results, we have:

Lemma 6.3. *The poles and zeros of an* LC *impedance (admittance) functions are all on the $j\omega$-axis and are all simple.*

Because the degree of the numerator polynomial cannot differ by more than one from the degree of the denominator polynomial, and each is either even or odd, we have

Lemma 6.4. *An* LC *impedance (admittance) function must have a simple pole or zero at $s = \infty$ and a simple pole or zero at $s = 0$.*

We now state an important theorem about *LC* impedance functions:

Theorem 6.1. *The necessary and sufficient conditions that a rational function $F(s)$ be an impedance (admittance) function of an* LC *circuit are:*

1. *$F(s)$ is a positive real function;*
2. *$F(s)$ has poles only on the $j\omega$-axis; and*
3. *$F(s)$ has either a pole or zero at $s = \infty$ and it has a pole or zero at $s = 0$.*

Proof. We have already proved the necessity part of the theorem in the lemmas. For the sufficiency part, let $F(s)$ be a rational function that satisfies the three conditions. Then since $F(s)$ is a positive real function, its partial fraction expansion at the poles on the $j\omega$-axis, including poles at ∞ and zero, all have positive real residues, as follows.

$$F(s) = r_\infty s + \frac{r_0}{s} + \sum_k \frac{2r_k s}{s^2 + \omega_k^2}, \tag{6.3}$$

where r_k are the residues at $s = j\omega_k$, r_∞ is the residue at a pole at $s = \infty$ and r_0 is the residue at a pole at $s = 0$, if any. We recognize that in Eq. (6.3), if $F(s)$ is an impedance function, it has a realization consisting of a series connection of an inductor, a capacitor, and a number of parallel *LC* circuits, as shown in Fig. 6.1, in which the element values are:

$$L_\infty = r_\infty, C_0 = 1/r_0, C_k = 1/2r_k, L_k = 2r_k/\omega_k^2. \tag{6.4}$$

If $F(s)$ is an admittance, its realization is shown in Fig. 6.2 with element values

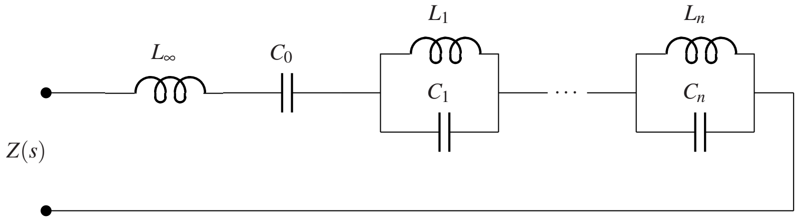

Fig. 6.1 A partial fraction realization of an *LC* impedance function.

given by

$$C_\infty = r_\infty, L_0 = 1/r_0, L_k = 1/2r_k, C_k = 2r_k/\omega_k^2. \tag{6.5}$$

So we have produced a circuit whose impedance or admittance is the given $F(s)$, and sufficiency is proved. □

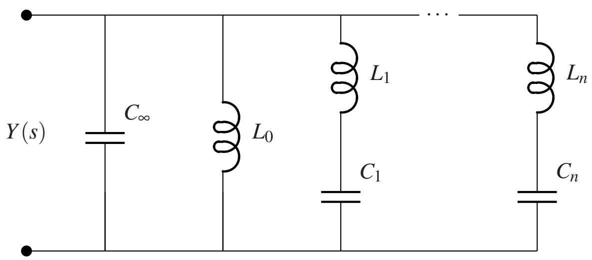

Fig. 6.2 A partial fraction realization of an *LC* admittance.

The circuit realizations of an *LC* impedance function shown in Figs. 6.1 and 6.2 obtained by partial fraction expansion of $Z(s)$ and its reciprocal $Y(s)$ are known as the Foster-I and Foster-II forms, respectively [26]. The residues are easily computed as follows.

$$r_\infty = \left.\frac{F(s)}{s}\right|_{s=\infty}, \tag{6.6}$$

$$r_0 = sF(s)|_{s=0}, \tag{6.7}$$

$$2r_k = \left.\frac{(s^2+\omega_k^2)F(s)}{s}\right|_{s^2=-\omega_k^2}. \tag{6.8}$$

6.1.2 Alternating poles and zeros

Additional properties of an *LC* impedance function can be deduced from its partial fraction expansion. Suppose $F(s) = Z(s)$ in Eq. (6.3). Let $Z(j\omega) = jX(\omega)$. The function $X(\omega)$ is called the *reactance* function of an *LC* impedance, given by

$$X(\omega) = r_\infty \omega - \frac{r_0}{\omega} + \sum_k \frac{2r_k\omega}{-\omega^2+\omega_k^2}. \tag{6.9}$$

It is an odd function of ω as expected. The slope of the function is

$$\frac{dX(\omega)}{d\omega} = r_\infty + \frac{r_0}{\omega^2} + \sum_k \frac{2r_k(\omega^2+\omega_k^2)}{(-\omega^2+\omega_k^2)^2}. \tag{6.10}$$

It is non-negative for all ω and we have

Theorem 6.2. *The reactance function of an* LC *impedance function has non-negative slope for all* ω.

As a result, we obtain further characterization:

Theorem 6.3. *The poles and zeros of an* LC *impedance (admittance) function alternate on the* $j\omega$*-axis.*

Proof. Let us write the impedance function in terms of its poles and zeros, as follows.

$$Z(s) = K\frac{(s^2+\omega_1^2)(s^2+\omega_3^2)\cdots(s^2+\omega_{2n+1}^2)}{s(s^2+\omega_2^2)(s^2+\omega_4^2)\cdots(s^2+\omega_{2n}^2)}, \tag{6.11}$$

where without loss of generality we assume it has a pole at $s=\infty$ and at $s=0$. Its reactance function is:

$$X(\omega) = -K\frac{(-\omega^2+\omega_1^2)(-\omega^2+\omega_3^2)\cdots(\omega^2+\omega_{2n+1}^2)}{\omega(-\omega^2+\omega_2^2)(-\omega^2+\omega_4^2)\cdots(-\omega^2+\omega_{2n}^2)}. \tag{6.12}$$

Imposing the condition that its slope cannot be negative, we conclude that two zeros or two poles cannot be adjacent and the poles and zeros must alternate.

As a consequence, in the general expression of an *LC* impedance function (6.11) the following must hold:

$$0 < \omega_1 < \omega_2 < \cdots < \omega_{2n} < \omega_{2n+1}\,. \tag{6.13}$$

The two Foster realizations of an *LC* impedance function are called *canonical* in a sense that the number of circuit parameters (element values) is precisely the same as the number of coefficients of the impedance function. Each realization requires the fewest number of elements. For example, $Z(s)$ of Eq. (6.11) has $2n+2$ parameters. Its two Foster realizations of Figs. 6.1 and 6.2 both have $2n+2$ elements.

6.1.3 Cauer realizations

There are two other canonical realizations. Let us write the impedance function of Eq. (6.11) as a ratio of two polynomials in descending order, assuming it has a pole at $s=\infty$:

$$Z(s) = \frac{b_n s^{2n+2} + b_{n-1}s^{2n} + \cdots + b_1 s^2 + b_0}{a_n s^{2n+1} + a_{n-1}s^{2n-1} + \cdots + a_1 s^3 + a_0 s}\,. \tag{6.14}$$

The pole at $s=\infty$ has a residue $r_\infty = b_n/a_n$. By the removal theorem of the last chapter, we can take it out and leave a positive real remainder $Z_1(s)$. Moreover, by Eq. (6.3), $Z_1(s)$ is an *LC* impedance function of lower order by one and it has a zero at $s=\infty$. Its reciprocal $Y_1(s) = 1/Z_1(s)$ has a pole there. Remove it and again we have a positive real remainder that is an *LC* admittance function with a zero at $s=\infty$. Take the reciprocal and remove the pole at infinity. This process is repeated until the remainder is zero. We illustrate the expansion with an example.

Example 6.1. Obtain an expansion of the following *LC* impedance function about $s=\infty$.

$$Z(s) = \frac{s^4+4s^2+3}{s^3+2s}\,. \tag{6.15}$$

Removal of the pole at $s=\infty$ gives us

$$Z(s) = s + \frac{2s^2+3}{s^3+2s} = s + Z_1(s)\,. \tag{6.16}$$

Reciprocate the remainder and take out its pole at $s=\infty$.

$$Y_1(s) = \frac{1}{Z_1(s)} = \frac{1}{2}s + \frac{s/2}{2s^2+3}\,. \tag{6.17}$$

Continuing, we obtain a *continued fraction expansion* of $Z(s)$ about $s=\infty$:

$$Z(s) = s + \cfrac{1}{\cfrac{s}{2} + \cfrac{1}{4s + \cfrac{1}{s/6}}}. \tag{6.18}$$

The continued fraction (6.18) can be realized as the impedance of a ladder circuit shown in Fig. 6.3**a**. It is called the Cauer I realization [16]. It has as many circuit parameters as the number of coefficients in the impedance function. So it is a canonical realization. The expansion can be obtained by a series of long divisions. At each

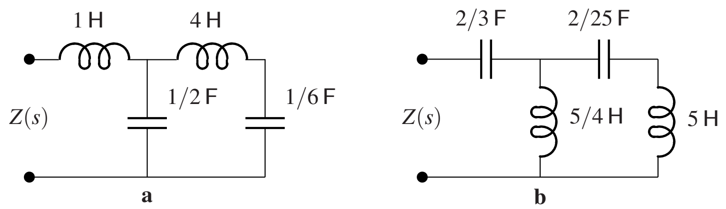

Fig. 6.3 **a** Cauer I realization of continued fraction expansion about $s = \infty$. **b** Cauer II realization about $s = 0$.

step, we extract a pole at infinity and get a remainder polynomial. At the next step, the remainder becomes the divisor and the previous divisor becomes the dividend. We continue until the remainder is zero. The single term quotients are alternately the impedances and admittances of the branches of a ladder.

If an LC impedance function has a pole at $s = 0$ (if not, its reciprocal does), it can be expanded in a continued fraction about $s = 0$. As higher order terms are insignificant compared to lower order terms near $s = 0$, we write the numerator and denominator polynomials in ascending order and extract a pole at $s = 0$ at each stage of the expansion.

Example 6.2. The continued fraction expansion of $Z(s)$ of the last example about $s = 0$ yields:

$$Z(s) = \frac{3}{2s} + \cfrac{1}{\cfrac{4}{5s} + \cfrac{1}{\cfrac{25}{2s} + \cfrac{1}{\cfrac{1}{5s}}}}. \tag{6.19}$$

Its impedance realization, called Cauer II, is shown in Fig. 6.3**b**. Again, it is a canonical realization.

6.1.4 Summary

An LC impedance function is a positive real function whose poles and zeros alternate on the $j\omega$-axis and it is an odd function of s. Given an LC impedance function

$Z(s)$, there are four canonical realizations: (1) Foster I, which is a partial fraction expansion of $Z(s)$; (2) Foster II, which is a partial expansion of $1/Z(s)$; (3) Cauer I, which is a continued fraction expansion about $s = \infty$; and (4) Cauer II, which is a continued fraction expansion about $s = 0$. The number of elements in each equals the number of coefficients of the given function $Z(s)$.

It must be noted that we can begin with a Foster expansion for one or two terms and switch to a Cauer expansion for the remainder, or vice versa. So there are many possible realizations for the same impedance function. However, none can recover the non-series-parallel circuit of Fig. 6.4, which has an LC impedance given by

$$Z(s) = \frac{2s^4 + 5s^2 + 1}{s^5 + 5s^3 + 2s} = \frac{1}{2s} + \frac{0.4468\,s}{s^2 + 0.4384} + \frac{1.0532\,s}{s^2 + 4.5615}\,. \tag{6.20}$$

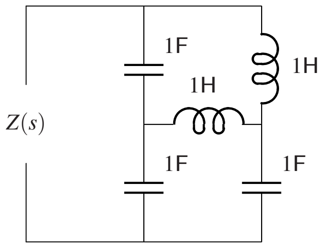

Fig. 6.4 A non-series-parallel circuit whose impedance cannot be recovered from standard synthesis methods.

6.2 *RC* impedance function

We begin again with the general expression of the impedance of a circuit in terms of energy. In this case, the inductive term is absent and we have

$$Z_{RC}(p) = \sum_k R_k I_{R_k} I_{R_k}^* + \frac{1}{p} \sum_k \frac{1}{C_k} I_{C_k} I_{C_k}^*\,, \tag{6.21}$$

where $p = r + jq$ is the complex frequency variable with real part r and imaginary part q. Let us take the general expression for an LC impedance from Eq. (6.1) and divide it by s. We get

$$\frac{Z_{LC}(s)}{s} = \sum_k L_k I_{L_k} I_{L_k}^* + \frac{1}{s^2} \sum_k \frac{1}{C_k} I_{C_k} I_{C_k}^*\,. \tag{6.22}$$

It is an even function of s and it does not have a pole at $s = \infty$. Comparing it with $Z_{RC}(p)$, we see they are identical in form provided we make the transformation:

$$p = s^2 . \tag{6.23}$$

In other words, $Z_{RC}(p)$ has the same functional properties as $Z_{LC}(s)/s$. Moreover the transformation $p = s^2$ maps the right-half s plane onto the entire p plane. In particular, the $j\omega$-axis is mapped onto the negative real axis of the p plane. The poles of $Z_{LC}(s)$ on the $j\omega$-axis are now poles on the negative r axis in the p plane. $Z_{RC}(p)$ has the same partial fraction expansion as $Z_{LC}(s)/s$.

6.2.1 Necessary and sufficient conditions

From the discussion above, we conclude that

Theorem 6.4. *The necessary and sufficient conditions that a rational function $F(s)$ be the impedance function of an* RC *circuit are:*

1. *$F(s)$ is a positive real function;*
2. *$F(s)$ has poles only on the negative real axis, all simple with positive real residues; and*
3. *$F(\infty)$ is either a positive constant or zero.*

Proof. The necessity part has already been proved. Suppose $F(s)$ is a function that satisfies the three conditions. Expand $F(s)$ in partial fractions and we have

$$F(s) = r_\infty + \frac{r_0}{s} + \sum_k \frac{r_k}{s + s_k} , \tag{6.24}$$

which can be realized as an impedance consisting of a series connections of a resistor of resistance $R_\infty = r_\infty$, a capacitor of capacitance $C_0 = 1/r_0$, and a number of parallel RC circuits with $C_k = 1/r_k$ and $R_k = 1/(C_k s_k)$, as shown in Fig. 6.5.

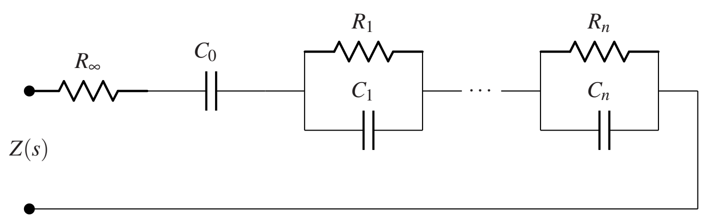

Fig. 6.5 Foster I realization of an *RC* impedance function.

6.2.2 Foster realizations

The realization of the partial fraction expansion of an *RC* impedance of Fig. 6.5 is called *RC Foster I* and it is canonical [26]. Next we consider an *RC* admittance function. It has a general form:

$$Y_{RC}(s) = \sum_k G_k V_{G_k} V_{G_k}^* + s \sum_k C_k V_{C_k} V_{C_k}^* . \tag{6.25}$$

Comparing it with $Z_{RC}(s)$, we see that $Y_{RC}(s)$ has the same form as $sZ_{RC}(s)$ and we have

Theorem 6.5. *The necessary and sufficient conditions that a rational function $F(s)$ be the admittance function of an* RC *circuit are:*

1. *$F(s)$ is a positive real function;*
2. *$F(s)$ has poles only on the negative real axis, all simple;*
3. *$F(s)/s$ has positive real residues at its poles; and*
4. *$F(0)$ is either a positive constant or zero.*

We will omit the proof. The function $F(s)$ can be realized as an *RC* admittance as it has an expansion given below.

$$Y_{RC}(s) = r_0 + sr_\infty + \sum_k \frac{sr_k}{s+s_k} . \tag{6.26}$$

The realization is shown in Fig. 6.6, known as Foster II. The element values are:

$$G_0 = r_0, C_\infty = r_\infty, G_k = r_k, C_k = G_k/s_k . \tag{6.27}$$

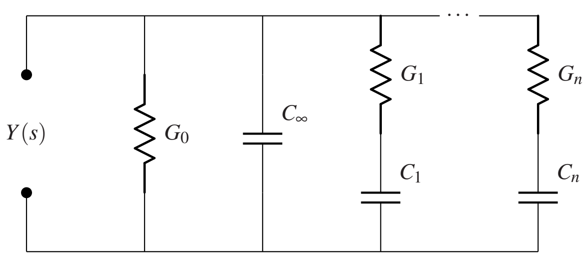

Fig. 6.6 Foster II realization of an *RC* admittance function.

6.2.3 Alternating poles and zeros

As in the case of an *LC* impedance functions, the poles and zeros of an *RC* impedance or admittance function alternate on the negative real axis. Consider the partial fraction expansion of $Z_{RC}(s)$ on the real axis $s = \sigma$:

$$Z_{RC}(\sigma) = r_\infty + \frac{r_0}{\sigma} + \sum_k \frac{r_k}{\sigma + s_k} . \tag{6.28}$$

Its derivative is non-positive for all σ. Since the poles of both $Z_{RC}(s)$ and $Y_{RC}(s)$ are simple, we conclude

Theorem 6.6. *The poles and zeros of an* RC *impedance or admittance function alternate on the negative real axis.*

As a consequence, an *RC* impedance function has the following general expression in terms of its poles and zeros:

$$Z_{RC}(s) = K \frac{(s+s_2)(s+s_4)\cdots(s+s_{2n})}{(s+s_1)(s+s_3)\cdots(s+s_{2n-1})} , \tag{6.29}$$

in which

$$0 \le s_1 < s_2 < \cdots < s_{2n} \le \infty \tag{6.30}$$

with the understanding that when $s_1 = 0$ we have a pole at $s = 0$ and when $s_{2n} = \infty$ we have a zero at $s = \infty$. The general expression of $Y_{RC}(s)$ is the reciprocal of $Z_{RC}(s)$.

6.2.4 Cauer realizations

There is one more property worthy of noting. Let us look at the real part of an *RC* impedance function on the $j\omega$-axis. From the partial fraction expansion, we get

$$\Re Z(j\omega) = R_\infty + \sum_k \frac{r_k s_k}{\omega^2 + s_k^2} . \tag{6.31}$$

It follows that

Theorem 6.7. *The minimum real part of* $Z_{RC}(j\omega)$ *is* $R_\infty = Z_{RC}(j\infty)$.

Similarly, we have

Theorem 6.8. *The minimum real part of* $Y_{RC}(j\omega)$ *is* $G_0 = Y_{RC}(0)$.

We will make use of these theorems in the continued fraction expansion of $Z_{RC}(s)$ and $Y_{RC}(s)$.

Let $Z(s)$ be an *RC* impedance function that we wish to realize. If $Z(\infty) > 0$ we remove a resistor with resistance $Z(\infty)$, leaving a remainder that by the removal theorem of the last chapter is positive real. Moreover, it has a zero at $s = \infty$. Invert the remainder. The admittance function has a pole there. We remove it as a capacitor, again leaving a positive real remainder which is an *RC* admittance. Invert it. Remove the minimum real part at $s=\infty$, and so on. We illustrate the process with an example.

Example 6.3. Expand the following impedance function in continued fraction about $s=\infty$.

$$Z(s)=\frac{120s^2+96s+9}{120s^2+36s+1}\,. \tag{6.32}$$

Note that the poles and zeroes alternate on the negative real axis. Following the procedure outlined above, we get:

$$Z(s)=1+\cfrac{1}{2s+\cfrac{1}{3+\cfrac{1}{4s+\cfrac{1}{5}}}}\,. \tag{6.33}$$

The realization is an *RC* ladder shown in Fig. 6.7**a**. The circuit is called Cauer I and it is canonical [16].

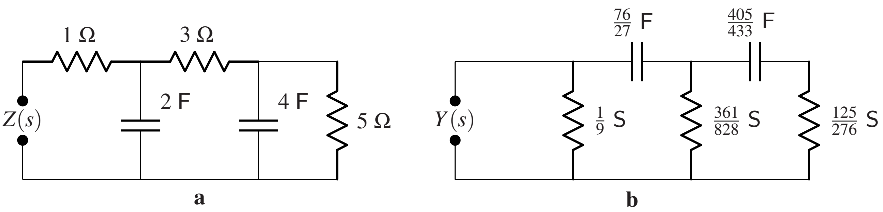

Fig. 6.7 **a** RC Cauer I realization. **b** RC Cauer II realization.

In a similar manner, if we expand $Y(s)=1/Z(s)$ of the last example in continued fraction about $s=0$, where the minimum real part of $Y(s)$ occurs, we obtain the Cauer II realization shown in Fig. 6.7**b**.

6.3 *RL* impedance function

Starting with the general expression of the impedance of an *RL* circuit in terms of energy:

$$Z_{RL}(s)=\sum_k R_k I_{R_k} I^*_{R_k}+s\sum_k L_k I_{L_k} I^*_{K_k}\,, \tag{6.34}$$

we see it has the same form as $sZ_{RC}(s)$. It follows that $Z_{RL}(s)$ has the same properties as $Y_{RC}(s)$ and $Y_{RL}(s)$ has the same properties as $Z_{RC}(s)$. We will not repeat the derivations and simply summarize the results as follows.

6.3.1 Analytic properties

Theorem 6.9. *The necessary and sufficient conditions that a rational function $F(s)$ be the impedance function of an* RL *circuit are:*

1. *$F(s)$ is a positive real function;*
2. *$F(s)$ has poles only on the negative real axis, all simple;*
3. *$F(s)/s$ has positive real residues at its poles; and*
4. *$F(0)$ is either a positive constant or zero.*

Theorem 6.10. *The necessary and sufficient conditions that a rational function $F(s)$ be the admittance function of an* RL *circuit are:*

1. *$F(s)$ is a positive real function;*
2. *$F(s)$ has poles only on the negative real axis, all simple with positive real residues; and*
3. *$F(\infty)$ is either a positive constant or zero.*

Theorem 6.11. *The poles and zeros of an* RL *impedance or admittance function are simple and alternate on the negative real axis.*

Theorem 6.12. *The minimum real part of $Z_{RL}(j\omega)$ is $Z_{RL}(0)$.*

Theorem 6.13. *The minimum real part of $Y_{RL}(j\omega)$ is $Y_{RL}(j\infty)$.*

6.3.2 Realizations

The two Foster realizations are shown in Fig. 6.8 while the two Cauer realizations in Fig. 6.9.

6.4 Remarks

Every two-element-kind impedance function has at least four realizations, two Foster and two Cauer forms. All are based on successive removal of an impedance while leaving a remainder that is a positive real function of lower order. But this procedure inevitably leads to a series-parallel realization. No known synthesis method exists which will lead to a general non-series-parallel circuit. For example, the following function

$$F(s) = \frac{(2s+3)(2s^2+8s+1)}{(s+1)(4s^2+15s+1)} = \frac{(s+0.1292)(s+1.5)(s+3.8708)}{(s+0.0679)(s+1)(s+3.6821)} \tag{6.35}$$

is the input impedance of the *RC* circuit of Fig. 6.10. Given $F(s)$, we do not know how to expand or unravel it to recover the circuit. The difficulty seems to be that we

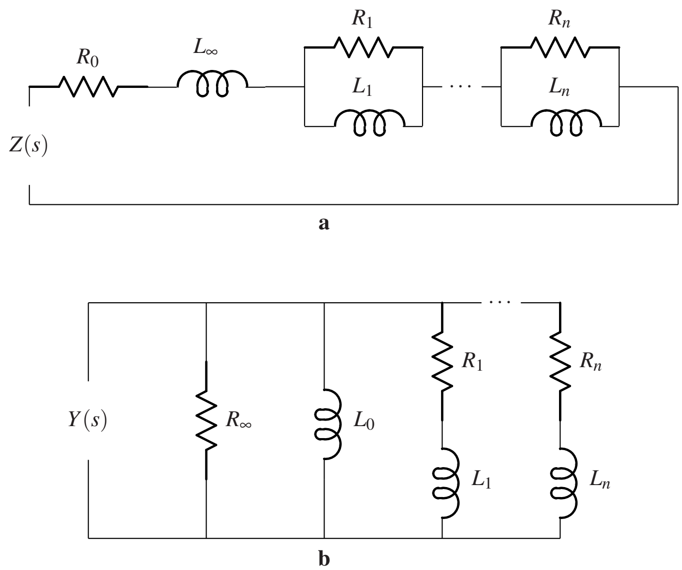

Fig. 6.8 **a** RL Foster I realization. **b** RL Foster II realization.

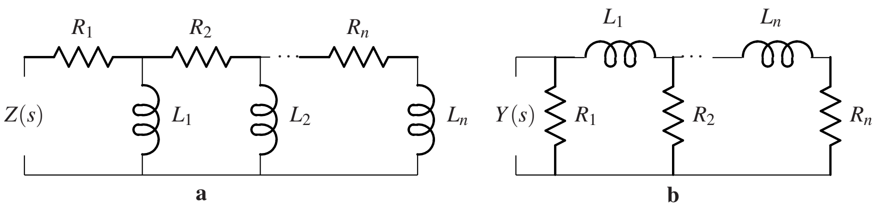

Fig. 6.9 **a** RL Cauer I realization. **b** RL Cauer II realization.

do not have a simple algebraic, possibly recursive, description of non-series-parallel circuits. An attempt to overcome this difficulty was reported in [37].

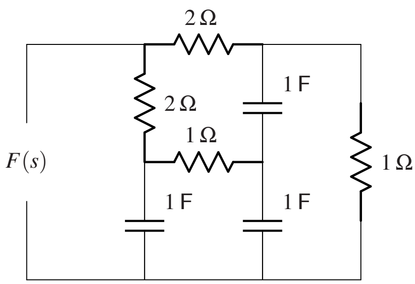

Fig. 6.10 A non-series-parallel *RC* circuit whose impedance cannot be recovered from standard synthesis methods.

Problems

6.1. Obtain the four canonical realizations of each of the following impedance functions.

$$(a)\ Z(s) = \frac{9s^4 + 10s^2 + 1}{4s^3 + s}\,.$$
$$(b)\ Z(s) = \frac{4s^3 + s}{9s^4 + 10s^2 + 1}\,.$$
$$(c)\ Z(s) = \frac{s^3 + 100s}{s^4 + 250s^2 + 5625}\,.$$
$$(d)\ Z(s) = \frac{s^4 + 250s^2 + 5625}{s^3 + 100s}\,.$$

6.2. Without actually expanding the functions, sketch the four canonical realizations of the following impedance functions.

$$(a)\ Z(s) = \frac{(s^2 + a_1^2)(s^2 + a_3^2)}{s(s^2 + a_2^2)(s^2 + a_4^2)}\,, \quad a_1 < a_2 < a_3 < a_4\,.$$
$$(b)\ Z(s) = \frac{s(s^2 + a_2^2)(s^2 + a_4^2)}{(s^2 + a_1^2)(s^2 + a_3^2)}\,.$$

6.3. Show that the impedance across $A - B$ of the circuit of Fig. 6.11 is

$$Z(s) = \frac{2s^5 + 11s^3 + 8s}{(s^2+1)(s^4 + 7s^2 + 4)} = \frac{2s(s^2 + 0.8625)(s^2 + 4.6375)}{(s^2 + 0.6277)(s^2 + 1)(s^2 + 6.3721)}\,.$$

Suppose $Z(s)$ is given. How can you recover this circuit from $Z(s)$? Are there other realizations besides Foster and Cauer?

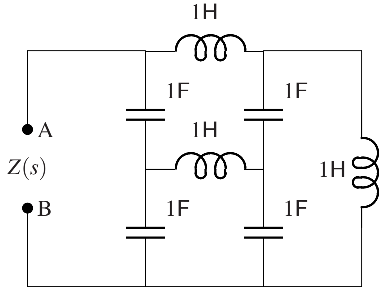

Fig. 6.11 A non-series-parallel LC circuit. Given the circuit, we can find the impedance $Z(s)$. Given $Z(s)$, we do not know how to recover the circuit

6.4. In the circuit of Fig. 6.12, find the impedance $Z(s)$ for values of the mutual inductance $M = 1/4, 1/2, 3/4, \text{and} 1$. Show that the impedances are all LC impedances. (Use symbolic math.) Obtain the Foster I realization of each.

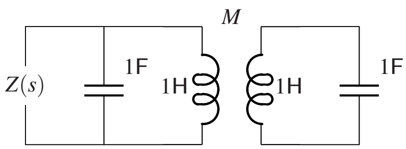

Fig. 6.12 Impedance of an LC circuit with coupled inductors

6.5. Which of the following impedance functions are RC or RL?

$$(a)\ Z(s) = \frac{s^2+8s+15}{s^2+6s+8}. \qquad (b)\ Z(s) = \frac{s^2+6s+8}{s^2+8s+15}.$$

$$(c)\ Z(s) = \frac{s^2+8s+15}{s^3+6s^2+8s}. \qquad (d)\ Z(s) = \frac{s^3+8s^2+15s}{s^2+6s+8}.$$

$$(e)\ Z(s) = \frac{s^2+3s+1}{s^2+4s+2}. \qquad (f)\ Z(s) = \frac{s^2+7s+10}{s^2+7s+12}.$$

6.6. Without actually expanding the functions, sketch the four canonical realizations of the following impedance functions.

$$(a)\ Z(s) = \frac{(s+a_1)(s+a_3)}{s(s+a_2)(s+a_4)}, \quad a_1 < a_2 < a_3 < a_4.$$

$$(b)\ Z(s) = \frac{s(s+a_2)(s+a_4)}{(s+a_1)(s+a_3)}.$$

$$(c)\ Z(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)}.$$

$$(d)\ Z(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)}.$$

6.7. Obtain the four canonical realizations of the following impedance function.

$$Z(s) = \frac{2s^2+6s+4}{8s^3+36s^2+46s+15}.$$

6.8. Obtain the four canonical realizations of the following impedance function.

$$Z(s) = \frac{8s^3+36s^2+46s+15}{2s^2+6s+4}.$$

6.9. Let us call the poles and zeroes of an impedance function ”critical frequencies.” Show that the first critical frequency from the origin of an *RC* impedance function is a pole and that of an *RL* impedance function is a zero. In addition, show that the last critical frequency of an *RC* impedance function is a zero and that of an *RL* impedance function is a pole. See Problem 6.6.

6.10. (a) Show that the transfer function of an *RC* circuit has all its poles on the negative real axis.
(b) Show that the eigenvalues of the state equations of an *RC* circuit must be non-negative and are all distinct.
(c) Show that the impulse response of an *RC* circuit must be a sum of decaying exponentials of distinct time constants.

6.11. In integrated circuit design, the interconnects are often modeled as *RC* transmission lines, which are then further simplified to become *RC* ladders of various number of sections that are connected at various nodes to represent branches of an interconnect. The system of interconnects is basically an *RC* circuit. What is the form of the time response if a step voltage is applied at one node and the response is taken at another node? What is the form of the transfer function?

6.12. Let an integrated circuit interconnect be represented by five sections of *RC* ladder with $R = 0.1\,\Omega$ and $C = 1\,\text{pF}$. Compute the step response. Repeat for ten sections. See the results in Fig. 6.13.

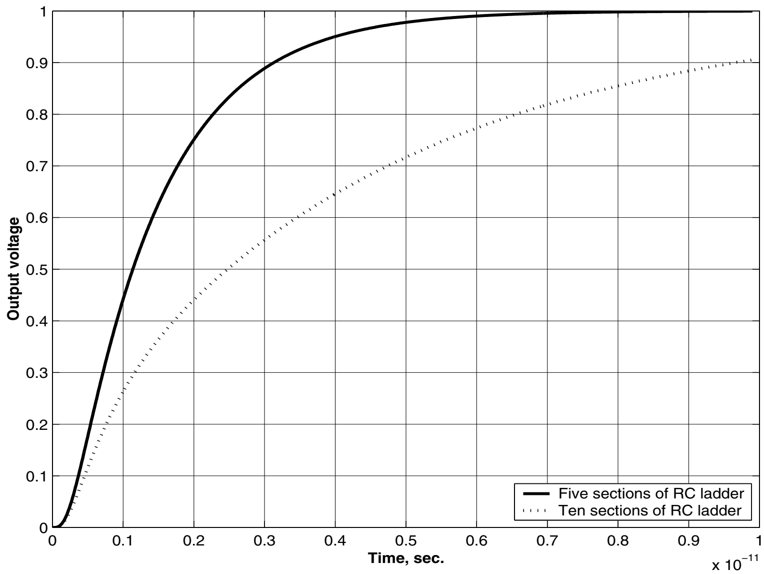

Fig. 6.13 Step response of lumped approximation of interconnect by sections of *RC* ladder

Chapter 7
Synthesis of *RLC* Impedances

We saw in the last chapter that the impedance function of an *LC*, *RC* or *RL* circuit is a positive real function whose poles and zeros are restricted to the imaginary or negative real axis. No such restrictions apply to an *RLC* impedance function. Its poles and zeros can be anywhere in the left-half s plane. We also noted that the synthesis of a two-element-kind impedance function amounts to partial fraction or continued fraction expansion and the resultant circuit is canonical. In general, no such expansion is known to exist for an *RLC* impedance function.

In this chapter we will present the Brune [12] realization of an *RLC* impedance function. He was first to demonstrate that every positive real function can be realized as the driving-point impedance of an *RLC* circuit, though ideal transformers are needed. It is a canonical realization in that the number of elements equals the number of coefficients of the impedance function. A transformerless realization was first reported by Bott-Duffin [9], but the number of elements required grows as $O(2^{n/2})$, where n is the order of the impedance function, and it is therefore not canonical. Given an arbitrary positive real function, finding a canonical realization without transformers is still an open question.

7.1 Brune synthesis

We begin with a positive real function $Z(s)$ and perform a series of preliminary steps to remove any and all poles and zeros on the $j\omega$ axis, as follows.

1. If $Z(s)$ has a pole at infinity, remove a series inductance and obtain a positive real remainder $Z_1(s)$.
2. If $Z(s)$ has a pole at zero, remove a series capacitor and obtain a p.r. remainder $Z_1(s)$.
3. If $Z(s)$ has a zero at infinity, remove a shunt capacitor from $Y(s) = 1/Z(s)$ and obtain a p.r. remainder $Y_1(s)$.
4. If $Z(s)$ has a zero at zero, remove a shunt inductor from $Y(s) = 1/Z(s)$ and obtain a p.r. remainder $Y_1(s)$.

5. If $Z(s)$ has poles at $s = \pm j\omega_0$, remove a parallel *LC* impedance and obtain a p.r. remainder $Z_1(s)$.
6. If $Z(s)$ has zeros at $s = \pm j\omega_0$, remove a series *LC* admittance from $Y(s) = 1/Z(s)$ and obtain a p.r. remainder $Z_1(s)$.

After all these removals, we are left with a remainder which does not have any poles or zeros on the $j\omega$ axis, including the points $s = \infty$ and $s = 0$. Such an impedance function is called a *minimum reactance* and *minimum susceptance* function. Brune synthesis begins at this point.

Let $Z(s)$ be a minimum reactance, minimum susceptance positive real function. For example,

$$Z(s) = \frac{3s^2 + 3s + 6}{2s^2 + s + 2}\,. \tag{7.1}$$

1. Let ω_0 be such that

$$\min_{\omega} \Re[Z(j\omega)] = \Re[Z(j\omega_0)] = R_{min}\,. \tag{7.2}$$

So ω_0 is the frequency at which the minimum real part of $Z(j\omega)$ occurs. For the example under consideration,

$$\omega_0 = \sqrt{2}, \qquad R_{min} = 1\,.$$

2. Remove R_{min} to get a remainder $Z_1(s)$, which is positive real.

$$Z_1(s) = Z(s) - R_{min} = \frac{s^2 + 2s + 4}{2s^2 + s + 2}\,. \tag{7.3}$$

3. Since the real part of $Z_1(j\omega_0) = 0$, $Z_1(j\omega_0) = jX$. We now have two cases: (A) $X < 0$ and (B) $X > 0$.

7.1.1 Case A: $X < 0$

For the running example, we have

$$Z_1(j\omega_0) = -j\sqrt{2} = jX\,, \tag{7.4}$$

and we have case (A).

4. Set the reactance X to be the reactance of an inductor L_1 at $j\omega_0$:

$$jX = j\omega_0 L_1, \quad L_1 = -1\,. \tag{7.5}$$

The inductance is negative but we proceed anyway.

5. Remove L_1 from $Z_1(s)$ to obtain $Z_2(s)$.

$$Z_2(s) = Z_1(s) - sL_1 = \frac{2s^3+2s^2+4s+4}{2s^2+s+2} = \frac{(s^2+2)(2s+2)}{(2s^2+s+2)}. \tag{7.6}$$

$Z_2(s)$ is positive real since we have just added a positive real function $-sL_1$ to $Z_1(s)$, and it has a pole at $s=\infty$. Moreover, it has a pair of zeros at $s=\pm j\omega_0$ because we have removed the real and imaginary parts of $Z(j\omega_0)$ from $Z(s)$.
6. Invert $Z_2(s)$ and remove the poles at $\pm j\omega_0$ to get a remainder $Y_3(s)$.

$$Y_3(s) = \frac{1}{Z_2(s)} - \frac{s/L_2}{s^2+1/L_2C} = \frac{1}{Z_2(s)} - \frac{s/2}{s^2+2} = \frac{1}{2s+2}. \tag{7.7}$$

The second term is a series resonance circuit consisting of $L_2 = 2\,\mathrm{H}$ and $C_2 = 1/4\,\mathrm{F}$.
7. The remainder impedance $Z_3(s) = 1/Y_3(s)$ has a pole at $s=\infty$, because we created one there when we removed a negative inductor from $Z_1(s)$.

$$Z_3(s) = 2s+2 = sL_3 + Z_4(s), \tag{7.8}$$

with $L_3 = 2\,\mathrm{H}$.
8. We have just completed one cycle of the Brune synthesis. The same procedure is now applied to $Z_4(s)$ and we repeat it until we have a remainder which is a constant, i.e., a resistor. In the running example, $Z_4(s) = 2\Omega$ and the synthesized circuit is shown Fig. 7.1**a**. The three inductors can be replaced by two coupled inductors with unity coupling coefficient as shown in Fig. 7.1**b**.

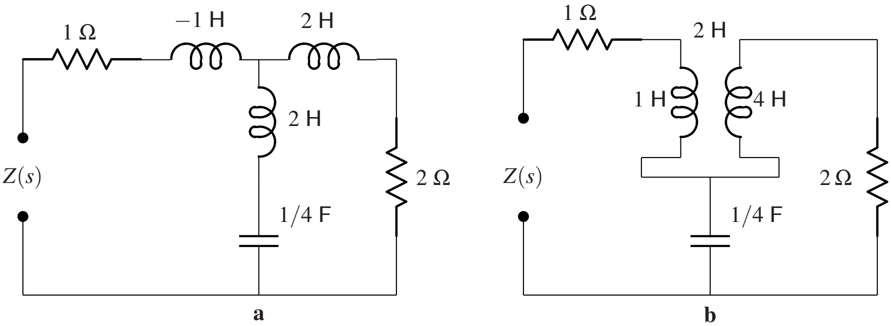

Fig. 7.1 **a** Brune realization of a minimum-reactance and minimum susceptance positive real function. **b** Replacing the three inductors with unity-coupled inductors

7.1.2 Ideal transformer

Brune's strategy is to take a minimum-reactance and minimum-susceptance function and deliberately create a pair of zeros on the $j\omega$ axis. The zeros are removed to leave a remainder which is two degrees lower than the original function. The process is repeated until we have a constant remainder. To create the zeros, we add an inductance, thereby creating a pole at infinity, which is later removed. The resulting circuit contains a negative inductance. However, the three inductors are equivalent to an ideal transformer, as shown in what follows.

Consider $Z_3(s)$, the remainder after the zeros are removed. It has a pole at $s = \infty$ whose residue is found from the following:

$$\lim_{s\to\infty} \frac{Z_3(s)}{s} = \lim_{s\to\infty} \frac{1/s}{\dfrac{1}{Z_1(s) - sL_1} - \dfrac{1}{sL_2 + \dfrac{1}{sC}}} = \frac{-L_1L_2}{L_1 + L_2} = L_3\,. \tag{7.9}$$

Equation (7.9) shows that

$$L_1L_2 + L_2L_3 + L_3L_1 = 0 \tag{7.10}$$

and the inductances are not independent.

Let us look at the ”Tee” of the three inductors which has a terminal characterization given as

$$\begin{aligned} V_1 &= s(L_1 + L_2)I_1 + sL_2I_2\,, \\ V_2 &= sL_2I_1 + s(L_2 + L_3)I_2\,. \end{aligned} \tag{7.11}$$

The ”Tee” is equivalent to and can be replaced by two coupled inductors with primary inductance L_A, secondary inductance L_B and mutual inductance M, given by

$$L_A = L_1 + L_2 \qquad L_B = L_2 + L_3 \qquad M = L_2\,. \tag{7.12}$$

The coupling coefficient is

$$k = \frac{|M|}{\sqrt{L_AL_B}} = \frac{|L_2|}{\sqrt{(L_1 + L_2)(L_2 + L_3)}}\,, \tag{7.13}$$

which is equal to unity if and only if the inductances satisfy Eq. (7.10). In fact, under this condition, the terminal characterization becomes linearly dependent and the voltages and the currents are related by, respectively:

$$V_2 = \frac{M}{L_A}V_1\,, \qquad I_2 = \frac{L_A}{M}I_1\,. \tag{7.14}$$

So the coupled inductors become an ideal transformer of turns ratio M/L_A.

7.1.3 Case B: $X > 0$

Consider a second example. Let $Z(s)$ be a minimum reactance, minimum susceptance positive real function:

$$Z(s) = \frac{2s^2+2s+5}{s^2+s+4}\,. \tag{7.15}$$

The minimum resistance and ω_0 are found to be

$$\omega_0 = \sqrt{2}\,, \qquad R_{min} = 1\,.$$

Remove R_{min} to get $Z_1(s)$.

$$Z_1(s) = Z(s) - R_{min} = \frac{s^2+s+1}{s^2+s+4}\,. \tag{7.16}$$

The reactance at $j\omega_0$ is

$$Z_1(j\omega_0) = j\frac{\sqrt{2}}{2} = jX\,, \tag{7.17}$$

and we have case (B).

If we take out an inductor $L_1 = X/\omega_0 = 1/2 > 0$ from $Z_1(s)$, the remainder $Z_2(s)$ is not a positive real function. Instead, we proceed on an admittance basis.

Let $Y_1(s) = 1/Z_1(s)$. Let $Y_1(j\omega_0) = 1/(jX) = jB$. Since $X > 0$, $B < 0$. We take out a capacitance

$$C_1 = \frac{B}{\omega_0} = -1 \tag{7.18}$$

from $Y_1(s)$ and get a remainder

$$Y_2(s) = Y_1(s) - sC_1 = \frac{(s^2+2)(s+2)}{s^2+s+1}\,. \tag{7.19}$$

We will take care of the negative capacitance later. The important thing to note is that $Y_2(s)$ is p.r. and it has zeros at $s = \pm j\omega_0$. Let $Z_2(s) = 1/Y_2(s)$. Remove its poles at $s = \pm j\omega_0$ and get a remainder:

$$Z_3(s) = Z_2(s) - \frac{1}{sC_2 + \dfrac{1}{sL}} = \frac{1}{2s+4}\,, \tag{7.20}$$

which is p.r., and $Y_3(s) = 1/Z_3(s)$ has a pole at $s = \infty$. Remove a capacitor $C_3 = 2\,\mathrm{F}$ and get a remainder

$$Y_4(s) = Y_3(s) - sC_3 = 4\,, \tag{7.21}$$

which is a constant.

The circuit realization is shown in Fig. 7.2. We will show that the Π of capacitors (together with L) can be converted to a "Tee" of inductors (together with C), as follows. With reference to Fig. 7.3, the two-port of the Π of capacitors can be

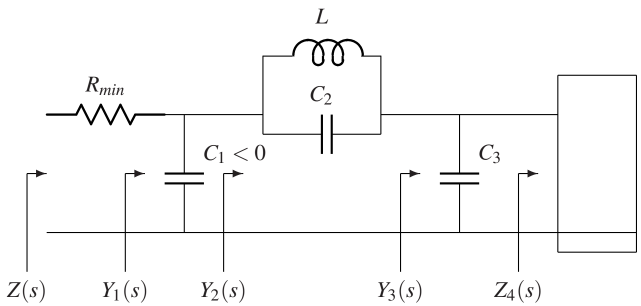

Fig. 7.2 Brune synthesis on an admittance basis

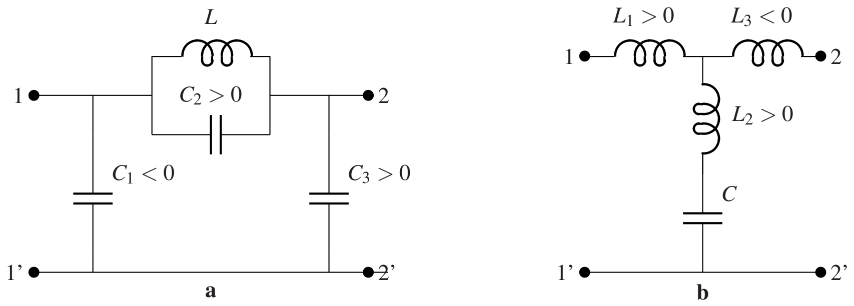

Fig. 7.3 Brune cycle on an admittance basis with a negative capacitance **a** is equivalent to a cycle on an impedance basis with a negative inductance **b**

described by its open-circuit impedance parameters:

$$z_{11}(s) = \frac{(C_2+C_3)Ls}{(C_1+C_3)} + \frac{1}{(C_1+C_3)s}; \tag{7.22}$$

$$z_{21}(s) = z_{12}(s) = \frac{C_2Ls}{(C_1+C_3)} + \frac{1}{(C_1+C_3)s}; \tag{7.23}$$

$$z_{22}(s) = \frac{(C_2+C_1)Ls}{(C_1+C_3)} + \frac{1}{(C_1+C_3)s}. \tag{7.24}$$

The open-circuit impedance parameters of the ”Tee” of inductors are:

$$z_{11}(s) = (L_1 + L_2)s + \frac{1}{Cs}; \tag{7.25}$$

$$z_{21}(s) = z_{12}(s) = L_2 s + \frac{1}{Cs}; \tag{7.26}$$

$$z_{22}(s) = (L_2 + L_3)s + \frac{1}{Cs}. \tag{7.27}$$

Equating the parameters, we obtain a set of relations among the capacitances and inductances, as follows.

$$L_1 = \frac{C_3}{C_1 + C_3} L > 0; \tag{7.28}$$

$$L_2 = \frac{C_2}{C_1 + C_3} L > 0; \tag{7.29}$$

$$L_3 = \frac{C_1}{C_1 + C_3} L < 0. \tag{7.30}$$

$$C = C_1 + C_3 > 0 \tag{7.31}$$

So the Π of capacitors together with L is equivalent to the "Tee" of inductors together with C. Moreover, the three inductances satisfy the equation:

$$L_1 L_2 + L_2 L_3 + L_3 L_1 = 0,$$

and the inductors can be replaced with two coupled coils with unity coupling coefficient, as shown in the final circuit of Fig. 7.4. This analysis shows that in the Brune synthesis, when $L_1 > 0$, we can proceed as the case of $L_1 < 0$, taking out L_1 and leaving a non-positive-real remainder $Z_2(s)$. At the end of the cycle, the remainder $Z_4(s)$ is two degrees lower and is a positive real function.

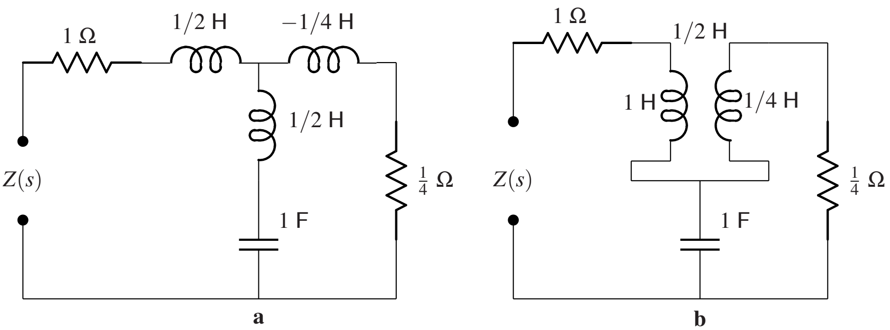

Fig. 7.4 Brune realization of Case (B): $X > 0$

7.1.4 Remarks

Each Brune section has four independent elements and the removal of a section reduces the order of the impedance by two. Adding the last resistor, we find the total number of elements equals the total number of coefficients of the impedance function. So the Brune realization is canonical.

Brune synthesis can be regarded as a realization of a given impedance function in a cascade of two-terminal pair sections, each consisting of a resistor, an ideal transformer and a shunt capacitance. Other points of view are reported in [5, 44].

7.2 Bott and Duffin synthesis

The Bott-Duffin synthesis is based on Richards' theorem [58]:

Theorem 7.1. *If $Z(s)$ is a positive real function, then $R(s)$ defined by*

$$R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)} \tag{7.32}$$

is also a positive real function for any real and positive constant k. Further, the degree of $R(s)$ is not higher than the degree of $Z(s)$.

To prove this theorem, we need the following lemmas.

Lemma 7.1. *Let $Z(s)$ be a positive real function and k be a positive real constant. Then*

$$W(s) = \frac{1 - \dfrac{Z(s)}{Z(k)}}{1 + \dfrac{Z(s)}{Z(k)}} \times \frac{k+s}{k-s} \tag{7.33}$$

is analytic in the right-half s plane and $|W(j\omega)| \leq 1$ for all ω.

Proof. The poles of $W(s)$ are the zeros of $1 + Z(s)/Z(k)$, which is a positive real function and has no zeros in the right-half s-plane. Therefore $W(s)$ is analytic in the right-half s plane. Second, let $Z(j\omega) = r(\omega) + jx(\omega)$. Consider

$$|W(j\omega)| = \left[\frac{(Z(k) - r(\omega))^2 + x(\omega)^2}{(Z(k) + r(\omega))^2 + x(\omega)^2}\right]^{1/2} . \tag{7.34}$$

Since $Z(s)$ is a positive real function, $r(\omega) \geq 0$ for all ω and $Z(k) \geq 0$. Therefore

$$|W(j\omega)| \leq 1 \quad \text{for all } \omega . \tag{7.35}$$

□

Lemma 7.2. *Let $W(s)$ be analytic in the right-half s-plane and $|W(j\omega)| \leq 1$. Then*

$$R(s) = \frac{1 - W(s)}{1 + W(s)} \tag{7.36}$$

is a positive real function.

Proof. Since $W(s)$ is analytic in the right-half s-plane, its maximum modulus occurs on the $j\omega$ axis. By hypothesis, $|W(j\omega)| \leq 1$. So $|W(s)| \leq 1$ for $\Re(s) \geq 0$. Second, let $W(s) = u(\sigma,\omega) + jv(\sigma,\omega)$. Consider the real part of $R(s)$:

$$\Re\{R(s)\} = \Re\left[\frac{1 - u(\sigma,\omega) - jv(\sigma,\omega)}{1 + u(\sigma,\omega) + jv(\sigma,\omega)}\right] = \frac{1 - |W(s)|^2}{(1 + u(\sigma,\omega))^2 + v^2(\sigma,\omega)}. \tag{7.37}$$

Since $|W(s)| \leq 1$ for $\Re(s) \geq 0$, $\Re\{R(s)\} \geq 0$ for $\Re(s) \geq 0$, and $R(s)$ is a positive real function. □

The proof of Theorem 7.1 follows if we note that

$$R(s) = \frac{1 - W(s)}{1 + W(s)} = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)} \tag{7.38}$$

and that $(s-k)$ is a common factor so that $deg\, R(s) \leq deg\, Z(s)$.

7.2.1 Synthesis procedure

Let $Z(s)$ be the impedance function that we wish to synthesize. We assume it has no poles or zeros on the $j\omega$ axis, including the origin and $s = \infty$ (if it does, we first remove them). We also assume it is minimum resistance, for we can always remove a resistor of value equal to the minimum resistance on the $j\omega$ axis. Suppose $R(s)$ is known (we will show later how it is found). Rewriting Eq. (7.32), we have

$$\begin{aligned} Z(s) &= \frac{kZ(k)R(s) + sZ(k)}{k + sR(s)} = \frac{kZ(k)R(s)}{k + sR(s)} + \frac{Z(k)s}{k + sR(s)} \\ &= \frac{1}{\dfrac{1}{Z(k)R(s)} + \dfrac{s}{kZ(k)}} + \frac{1}{\dfrac{k}{Z(k)s} + \dfrac{R(s)}{Z(k)}}, \end{aligned} \tag{7.39}$$

which can be realized as shown in Fig. 7.5. The two remainders

$$Z_1(s) = Z(k)R(s) \qquad Z_2(s) = \frac{Z(k)}{R(s)} \tag{7.40}$$

are positive real and their degrees are not higher than the degree of $Z(s)$.

Now choose a value for k such that $R(s)$ will have a pair of poles or zeros on the $j\omega$ axis. Remove them and leave two remainders with lower degrees. Since $Z(s)$

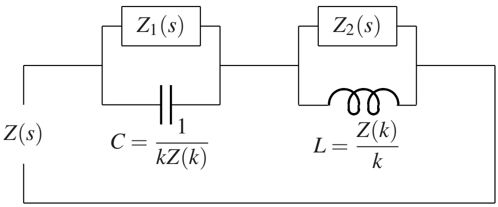

Fig. 7.5 The start of Bott-Duffin synthesis

is minimum resistance, there exists an ω_o such that $\Re Z(j\omega_o) = 0$. Let $Z(j\omega_o) = jX(\omega_o)$. We have two cases.

Case A: $X(\omega_o) > 0$. Choose k such that $R(s)$ will have a pair of zeros at $s = \pm j\omega_o$. From Eq. (7.32), we set

$$kZ(j\omega_o) - j\omega_o Z(k) = 0\,. \tag{7.41}$$

This gives us a polynomial in k. Solve for a positive real root and set it to k. Note that $k = \pm j\omega_o$ are roots of this equation and we can remove the factor $k^2 + \omega_o^2$ from the polynomial to simplify the root-finding procedure.

With k known, we have in Fig. 7.5

$$C = \frac{1}{kZ(k)} > 0\,, \quad L = \frac{Z(k)}{k} > 0\,. \tag{7.42}$$

$R(s)$ has a pair of zeros at $s = \pm j\omega_o$ by construction. The remainders can be expanded as

$$Y_1(s) = \frac{1}{Z_1(s)} = \frac{1}{Z(k)R(s)} = \frac{2k_1 s}{s^2 + \omega_o^2} + Y_3(s)\,, \tag{7.43}$$

$$Z_2(s) = \frac{Z(k)}{R(s)} = \frac{2k_2 s}{s^2 + \omega_o^2} + Z_4(s)\,. \tag{7.44}$$

They are positive real and are two degrees lower than $Z(s)$. The circuit realization is shown in Fig. 7.6, in which the elements have the following values:

$$L_1 = \frac{1}{2k_1}\,, \quad C_1 = \frac{2k_1}{\omega_o^2}\,, \quad L_2 = \frac{2k_2}{\omega_o^2}\,, \quad C_2 = \frac{1}{2k_2}\,. \tag{7.45}$$

This completes one cycle of Bott-Duffin realization. The same procedure is now applied to $Y_3(s)$ and $Z_4(s)$. Each cycle reduces the degree of $Z(s)$ by 2.

Case B: $X(\omega_o) < 0$. Choose k such that $R(S)$ will have a pair of poles at $s = \pm j\omega_o$. From Eq. (7.32), we set

$$kZ(k) - j\omega_o Z(j\omega_o) = 0\,, \tag{7.46}$$

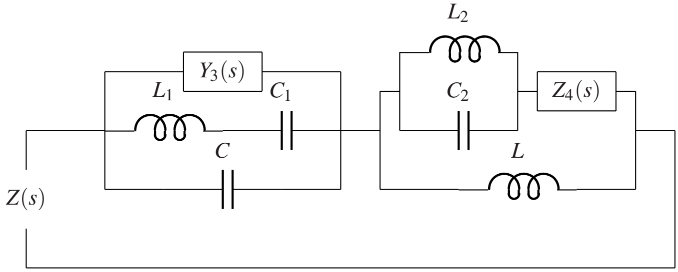

Fig. 7.6 One cycle of Bott-Duffin synthesis ($X(\omega_o) > 0$)

from which we determine k. $Z_1(s)$ and $Z_2(s)$ are then expanded as

$$Z_1(s) = Z(k)R(s) = \frac{2k_1 s}{s^2+\omega_o^2} + Z_3(s)\,, \tag{7.47}$$

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{R(s)}{Z(k)} = \frac{2k_2 s}{s^2+\omega_o^2} + Y_4(s)\,. \tag{7.48}$$

The circuit realization is shown in Fig. 7.7, in which the element values are given below.

$$C = \frac{1}{kZ(k)}\,, L = \frac{Z(k)}{k}\,, C_1 = \frac{1}{2k_1}\,, L_1 = \frac{2k_1}{\omega_o^2}\,, C_2 = \frac{2k_2}{\omega_o^2}\,, L_2 = \frac{1}{2k_2}\,. \tag{7.49}$$

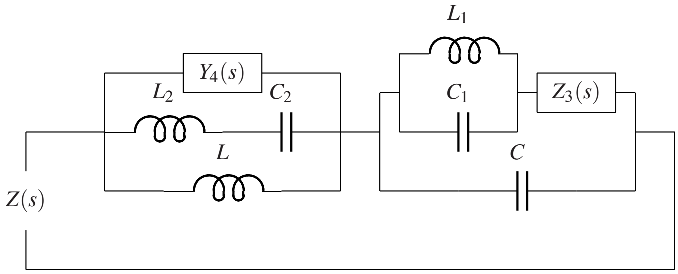

Fig. 7.7 One cycle of Bott-Duffin synthesis ($X(\omega_o) < 0$)

7.2.2 *Example*

Example 7.1. Find a Bott-Duffin realization of the following positive real function.

$$Z(s) = \frac{s^2 + 4s + 9}{s^2 + s + 1}. \tag{7.50}$$

We first note that $Z(s)$ is minimum reactance and minimum susceptance. Its real part along the $j\omega$ axis has a zero at $\omega_o = \sqrt{3}$. So

$$Z(j\omega_o) = -j2\sqrt{3} \tag{7.51}$$

and we have the second case. Setting

$$kZ(k) - j\omega_o Z(j\omega_o) = 0\,, \tag{7.52}$$

we find k to be

$$k = 2 \quad \text{and} \quad Z(k) = 3\,. \tag{7.53}$$

The circuit realization is shown in Fig. 7.8.

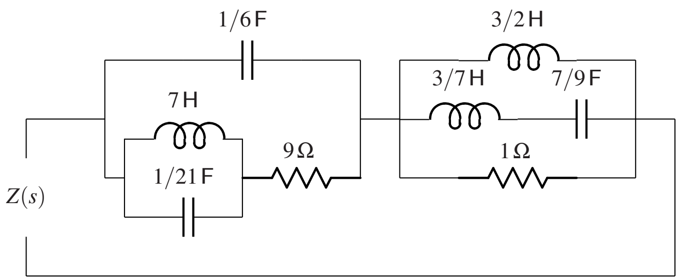

Fig. 7.8 Bott-Duffin realization of a second order impedance function

7.2.3 Remarks

In Fig. 7.5, $Z_1(s)Z_2(s) = Z^2(k)$, so $Z_1(s)$ and $Z_2(s)$ are dual impedances. Let n be the degree of $Z(s)$ and assume it is even. Then the total number of reactive elements (inductors and capacitors) is

$$E = 3(2^{n/2} + 2^{(n/2)-1} + \cdots + 2)\,. \tag{7.54}$$

In addition, we need $2^{n/2}$ resistors. For alternate approaches to Bott-Duffin synthesis, see [6, 32, 30].

7.3 Miyata synthesis

There are synthesis procedures that realize special cases of *RLC* impedance functions without transformers. For example, if $Z(s)$ is such that

$$\Re Z(j\omega) = \frac{a_n\omega^{2n} + a_{n-1}\omega^{2n-2} + \cdots + a_0}{Q(\omega^2)}, \tag{7.55}$$

in which all the coefficients $a_k \geq 0$, then $Z(s)$ can be realized as a series connection of impedances each of which has a real part of the form

$$\Re Z_k(j\omega) = \frac{a_k\omega^{2k}}{Q(\omega^2)}. \tag{7.56}$$

Such an impedance function can be realized as the input impedance of a lossless ladder circuit terminated in a resistor, as reported by Miyata [42]. As examples, suppose

$$\Re Z_1(j\omega) = \frac{1}{1+\omega^6} \tag{7.57}$$

$$\Re Z_2(j\omega) = \frac{\omega^2}{1+\omega^6} \tag{7.58}$$

$$\Re Z_3(j\omega) = \frac{1+4\omega^2}{1+\omega^6} \tag{7.59}$$

We can recover an impedance from its part as outlined in Sect. 4.5.3. For $Z_1(s)$ and $Z_2(s)$, we have

$$Z_1(s) = \frac{\frac{2}{3}s^2 + \frac{4}{3}s + 1}{s^3 + 2s^2 + 2s + 1} \tag{7.60}$$

$$Z_2(s) = \frac{\frac{1}{3}s^2 + \frac{2}{3}s}{s^3 + 2s^2 + 2s + 1} \tag{7.61}$$

By successive removal of zeros at infinity and at zero, we obtain the realizations of $Z_1(s)$ and $Z_2(s)$ shown in Fig. 7.9. As to $Z_3(s)$, it is a scaled sum of $Z_1(s)$ and $Z_2(s)$

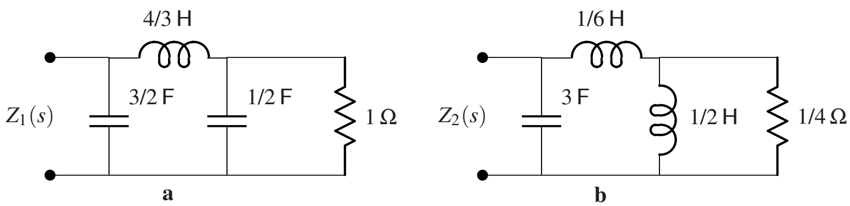

Fig. 7.9 Miyata realizations of two impedance functions

and its realization is given in Fig. 7.10.

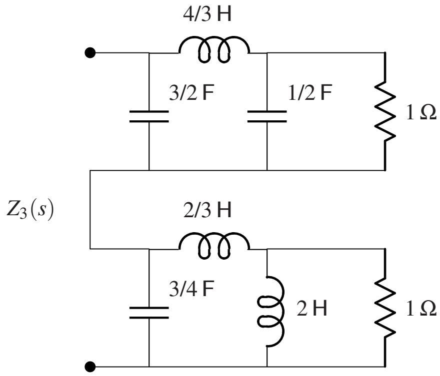

Fig. 7.10 Miyata realization of an impedance as a scaled sum of two impedances

7.4 General remarks

We have presented two synthesis methods to realize a general positive real function as the impedance of an *RLC* circuit. There is in fact a third general method known as Darlington synthesis [19]. He showed that a positive real function can be realized as the driving-point impedance of a lossless two-port terminated in a one-ohm resistor. The method is akin to Brune synthesis. In the latter, the circuit is a cascade of two-ports that each realizes a pair of zeros on the $j\omega$ axis that are created artificially. In Darlington's method, each section realizes zeros of the transfer function located anywhere. Since it is closely related to the synthesis of transfer functions, we will describe the method in a later chapter.

Darlington in effect demonstrated that any positive real function can be realized as an impedance of a circuit containing exactly one resistor, the rest being lossless elements. An interesting problem is to identify that subclass of positive real functions that can be realized as an impedance of a circuit containing exactly one inductor, the rest being resistors and capacitors. This turns out to be a very difficult problem, still not solved completely [46].

There exist many variations of Brune's and Bott-Duffin's methods [5, 6, 30, 31]. Each seeks to reduce the number of elements or to eliminate the need to use ideal transformers. It is interesting to note that since the publication of Bott-Duffin synthesis in 1949, there has not been any reported method that is fundamentally new.

Problems

7.1. Show that the impedance function that remains after each cycle of Brune realization is a positive real function.

7.2. Find a Brune realization of each of the following impedance functions.

$$(a) Z(s) = \frac{s^2+4s+9}{s^2+s+1}. \qquad (b) Z(s) = \frac{s^3+5s^2+10s+1}{s(s^2+s+1)}.$$

$$(c) Z(s) = \frac{2s^2+2s+5}{s^2+s+4}. \qquad (d) Z(s) = \frac{s^4+2s^3+3s^2+5s+1}{s^4+s^3+5s^2+s+4}.$$

7.3. Find a Brune realization of the following positive real function.

$$Z(s) = \frac{36s^4+45s^3+39s^2+12s+3}{16s^4+36s^3+25s^2+9s+1}.$$

7.4. Find a Bott-Duffin realization for each of the impedance function of Problem 7.2.

7.5. Not every positive real function need be synthesized by the Brune or Bott-Duffin method. Find a transformerless realization of each of the following positive real function.

$$(a)\ Z(s) = \frac{s^2+3s+1}{s^2+3s+2}. \qquad (b)\ Z(s) = \frac{s^2+3s+2}{s^2+3s+1}.$$

$$(c)\ Z(s) = \frac{s^2+3s+2}{s^2+2s+2}. \qquad (d)\ Z(s) = \frac{s^2+2s+2}{s^2+3s+2}.$$

$$(e)\ Z(s) = \frac{720s^3+544s^2+156s+18}{240s^3+308s^2+100s+3}. \qquad (e)\ Z(s) = \frac{240s^3+308s^2+100s+3}{720s^3+544s^2+156s+18}.$$

7.6. By repeated removals of poles and zeros on the $j\omega$ axis, realize each of the following positive real functions as the impedance of an *RLC* circuit without transformer.

$$(a)\ Z(s) = \frac{s^3+3s^2+2s+2}{s^2+3s+1}. \qquad (b)\ Z(s) = \frac{s^2+3s+1}{s^3+3s^2+2s+2}.$$

$$(c)\ Z(s) = \frac{s^4+2s^3+2s^2+2s+1}{s(s^3+s^2+2s+1)}. \qquad (d)\ Z(s) = \frac{s(s^3+s^2+2s+1)}{s^4+2s^3+2s^2+2s+1}.$$

7.7. The input impedance of the following circuit is:

$$Z(s) = \frac{2s^2+3s+3}{3s^2+3s+2}.$$

Suppose $Z(s)$ is given, can you recover the circuit? See Kim [37] for an exposition on synthesis of non-series-parallel impedance functions.

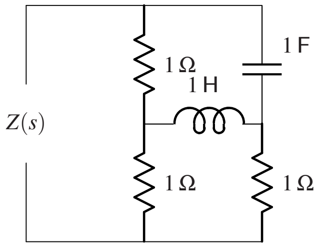

Fig. 7.11 A non-series-parallel *RLC* impedance

7.8. Apply the Miyata synthesis procedure to find a circuit realization by continued fraction expansion of each of the following impedance functions in the form of a *LC* ladder terminated in a resistor.

$$(a)\ Z(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)}. \qquad (b)\ Z(s) = \frac{s^2+2s+2}{s^3+3s^2+5s+3}.$$

7.9. Find an impedance function whose real part is each of the following. Synthesize the impedance function as the impedance of one or more lossless ladders each terminated in a resistor.

$$(a)\ \Re Z(j\omega) = \frac{1}{1+\omega^4}. \qquad (b)\ \Re Z(j\omega) = \frac{\omega^2}{1+\omega^4}.$$

$$(c)\ \Re Z(j\omega) = \frac{\omega^4}{1+\omega^4}. \qquad (d)\ \Re Z(j\omega) = \frac{2+\omega^2}{1+\omega^4}.$$

7.10. Find an impedance function whose real part is each of the following. Synthesize $Z(s)$ without transformer.

$$(a)\ \Re Z(j\omega) = \frac{2\omega^6+1}{\omega^6+1}. \qquad (b)\ \Re Z(j\omega) = \frac{2\omega^2+1}{\omega^6+1}.$$

Chapter 8
Scattering Matrix

We now turn to transfer functions. A transfer function describes certain input/output relation, and in circuits, it is usually the ratio of the voltage at an output terminal pair or output port, to the voltage of a voltage source connected across an input terminal pair or input port. The output port is usually terminated in some resistance or impedance, and the voltage source usually has an internal resistance or impedance associated with it. The situation is depicted in Fig. 8.1 for the case of resistive termination at both ports. The circuit in the middle is called a two-port. We define a

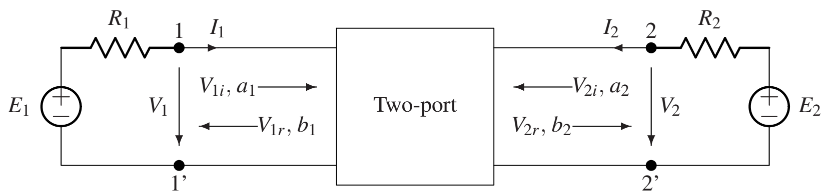

Fig. 8.1 A two-port with resistive terminations

voltage transfer function when port-2 is terminated in R_2 ($E_2 = 0$) as

$$H(s) = \left.\frac{V_2(s)}{E_1(s)}\right|_{E_2=0} \tag{8.1}$$

and ask: What are the necessary and sufficient conditions for a rational function $H(s)$ to be realizable as the voltage transfer function of a two-port comprising resistors, capacitors and inductors, and terminated in R_1 and R_2 at the two ports? To answer this question, we need to relate the transfer function to some physical properties of the two-port. For this purpose, it is most convenient to characterize the two-port by its *scattering matrix*.

8.1 Scattering matrix with resistive terminations

Scattering matrices were first used in the description of waveguide structures where it is natural to speak of incident and reflected or scattered waves at reference planes or boundaries. The concept is so useful that we have adopted it to describe two-port and multi-port circuits, especially at high frequencies where the measurement of voltage and current is all but impossible and must be made in terms of wave quantities.

8.1.1 Definition

We begin by noting that given any two quantities, we can always express them as the sum and difference of two other quantities. With reference to Fig. 8.1, let us write:

$$V_1(s) = V_{1i}(s) + V_{1r}(s) \qquad V_2(s) = V_{2i}(s) + V_{2r}(s)\,, \tag{8.2}$$

$$R_1 I_1(s) = V_{1i}(s) - V_{1r}(s) \qquad R_2 I_2(s) = V_{2i}(s) - V_{2r}(s)\,, \tag{8.3}$$

and the two pairs of quantities V_{1i}, V_{1r}, and V_{2i}, V_{2r} are found from

$$V_{1i}(s) = \frac{1}{2}(V_1(s) + R_1 I_1(s)) \qquad V_{2i}(s) = \frac{1}{2}(V_2(s) + R_2 I_2(s))\,, \tag{8.4}$$

$$V_{1r}(s) = \frac{1}{2}(V_1(s) - R_1 I_1(s)) \qquad V_{2r}(s) = \frac{1}{2}(V_2(s) - R_2 I_2(s))\,. \tag{8.5}$$

So far, these four quantities have no physical meaning. Consider the power taken by the circuit, viewed from port-1 with port-2 terminated in R_2:

$$P_1(j\omega) = \frac{1}{2}\Re[V_1(j\omega)I_1^*(j\omega)] = \frac{1}{2R_1}|V_{1i}(j\omega)|^2 - \frac{1}{2R_1}|V_{1r}(j\omega)|^2\,. \tag{8.6}$$

The first term can be interpreted as the power carried by an incident voltage V_{1i} and the second the power carried by a reflected voltage V_{1r}. Similarly, the power taken by the circuit viewed from port-2 with port-1 terminated in R_1 is

$$P_2(j\omega) = \frac{1}{2}\Re[V_2(j\omega)I_2^*(j\omega)] = \frac{1}{2R_2}|V_{2i}(j\omega)|^2 - \frac{1}{2R_2}|V_{2r}(j\omega)|^2\,. \tag{8.7}$$

Let us normalize the incident and reflected voltages by defining “incident wave” a_1 and “reflected wave” b_1 at port-1 as follows.

$$a_1(s) = \frac{1}{\sqrt{2R_1}}V_{1i}(s) = \frac{1}{2\sqrt{2R_1}}(V_1(s) + R_1 I_1(s))\,; \tag{8.8}$$

$$b_1(s) = \frac{1}{\sqrt{2R_1}}V_{1r}(s) = \frac{1}{2\sqrt{2R_1}}(V_1(s) - R_1 I_1(s))\,. \tag{8.9}$$

The physical significance of a_1 and b_1 is apparent when we note that

$$|a_1(j\omega)|^2 = \frac{1}{2R_1}|V_{1i}(j\omega)|^2\,; \tag{8.10}$$

$$|b_1(j\omega)|^2 = \frac{1}{2R_1}|V_{1r}(j\omega)|^2\,. \tag{8.11}$$

So $|a_1(j\omega)|^2$ and $|b_1(j\omega)|^2$ are the incident and reflected power, respectively, at port-1 when port-2 is terminated in R_2. Similarly we define

$$a_2(s) = \frac{1}{\sqrt{2R_2}}V_{2i}(s) = \frac{1}{2\sqrt{2R_2}}(V_2(s)+R_2I_2(s))\,, \tag{8.12}$$

$$b_2(s) = \frac{1}{\sqrt{2R_2}}V_{2r}(s) = \frac{1}{2\sqrt{2R_2}}(V_2(s)-R_2I_2(s))\,, \tag{8.13}$$

as the incident and reflected waves, respectively, at port-2 when port-1 is terminated in R_1. These quantities are shown in Fig. 8.1.

It is also useful to express the terminal voltages and currents in terms of the $a's$ and $b's$, as follows.

$$V_1 = \sqrt{2R_1}(a_1(s)+b_1(s)) \qquad V_2 = \sqrt{2R_2}(a_2(s)+b_2(s))\,; \tag{8.14}$$

$$I_1 = \sqrt{\frac{2}{R_1}}(a_1(s)-b_1(s)) \qquad I_2 = \sqrt{\frac{2}{R_2}}(a_2(s)-b_2(s))\,. \tag{8.15}$$

Since the two-port is linear, the $b's$ are related to $a's$:

$$\begin{bmatrix} b_1(s) \\ b_2(s) \end{bmatrix} = \begin{bmatrix} S_{11}(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} \begin{bmatrix} a_1(s) \\ a_2(s) \end{bmatrix}. \tag{8.16}$$

The matrix S

$$S(s) = \begin{bmatrix} S_{11}(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} \tag{8.17}$$

is called the *scattering matrix* of the two port and $S_{ij}(s)$ the scattering coefficients or parameters. Note that $S(s)$ is defined with respect to the terminating resistances R_1 at port-1 and R_2 at port-2.

The scattering parameters are computed and interpreted as follows. With reference to Fig. 8.1 and noting that the conditions, $a_2 = 0$, $V_2 = -R_2I_2$, and $E_2 = 0$, are equivalent, and similarly for port-1, we get:

$$S_{11}(s) = \left.\frac{b_1(s)}{a_1(s)}\right|_{a_2=0} = \left.\frac{V_{1r}(s)}{V_{1i}(s)}\right|_{V_2=-R_2I_2}, \tag{8.18}$$

$$S_{12}(s) = \left.\frac{b_1(s)}{a_2(s)}\right|_{a_1=0} = \left.\frac{\sqrt{R_2}V_{1r}(s)}{\sqrt{R_1}V_{2i}(s)}\right|_{V_1=-R_1I_1} = -2\sqrt{R_1R_2}\left.\frac{I_1}{E_2}\right|_{E_1=0}, \tag{8.19}$$

$$S_{21}(s) = \left.\frac{b_2(s)}{a_1(s)}\right|_{a_2=0} = \left.\frac{\sqrt{R_1}V_{2r}(s)}{\sqrt{R_2}V_{1i}(s)}\right|_{V_2=-R_2I_2} = -2\sqrt{R_1R_2}\left.\frac{I_2}{E_1}\right|_{E_2=0}, \tag{8.20}$$

$$S_{22}(s) = \left.\frac{b_2(s)}{a_2(s)}\right|_{a_1=0} = \left.\frac{V_{2r}(s)}{V_{2i}(s)}\right|_{V_1=-R_1I_1}. \tag{8.21}$$

The parameters $S_{11}(s)$ and $S_{22}(s)$ are called the *reflection coefficients* at port-1 and port-2, respectively, each being the ratio of the reflected wave to the incident wave. $S_{21}(s)$ indicates how much of the incident power at port-1 is transmitted to port-2. Similarly for $S_{12}(s)$. They are called the *transmission functions* of the two-port. The reflection coefficients and transmission functions have additional properties.

8.1.2 Reflection coefficient

From Eqs. (8.18), (8.8), and (8.9), and noting that V_1/I_1 is the input impedance $Z_{in(1)}$ looking into port-1 with port-2 terminated in R_2, we obtain

$$S_{11}(s) = \frac{Z_{in(1)} - R_1}{Z_{in(1)} + R_1}. \tag{8.22}$$

Similarly,

$$S_{22}(s) = \frac{Z_{in(2)} - R_2}{Z_{in(2)} + R_2}. \tag{8.23}$$

Since $Z_{in(1)}$ is a rational and positive real function, by Theorem (5.1), we have

Theorem 8.1. *The reflection coefficient $S_{11}(s)$ of an* RLC *two-port has the following properties:*

1. *$S_{11}(s)$ is real when s is real;*
2. *$S_{11}(s)$ is analytic in the right half s-plane;*
3. *$|S_{11}(j\omega)| \le 1$ for all ω, and $|S_{11}(s)| \le 1$ for $\Re(s) \ge 0$.*

So $S_{11}(s)$ maps the right half s-plane into a unit circle in the S_{11}-plane. The $j\omega$ axis is mapped onto the circumference. Similarly for $S_{22}(s)$.

Equation (8.22) establishes a relation between the input impedance and the reflection coefficient of a two-port. Given $S_{11}(s)$, $Z_{in(1)}$ is found from

$$Z_{in(1)} = R_1\,\frac{1+S_{11}(s)}{1-S_{11}(s)}. \tag{8.24}$$

This relation will be used in transfer function synthesis.

8.1.3 Transmission function

If the two-port is *RLC*, it is reciprocal, and in Eqs. (8.19) and (8.20), I_2/E_1 when $E_2 = 0$ equals I_1/E_2 when $E_1 = 0$. So we have

Theorem 8.2. *In an* RLC *two-port terminated in resistors, its transmission function from port-1 to port-2 is the same as that from port-2 to port-1, namely,* $S_{12}(s) = S_{21}(s)$.

Using Eqs. (8.13), (8.8) and (8.1), we can express $S_{21}(s)$ in terms of the two-port voltages:

$$S_{21}(s) = \left.\frac{b_2(s)}{a_1(s)}\right|_{V_2=-R_2I_2} = 2\sqrt{\frac{R_1}{R_2}}\left.\frac{V_2(s)}{E_1(s)}\right|_{E_2=0} = 2\sqrt{\frac{R_1}{R_2}}H(s)\,. \tag{8.25}$$

So $S_{21}(s)$ is proportional to the transfer function, whose poles, as noted in Observation 4.7, are the same as the zeros of the impedance seen by E_1 and are in the left half s-plane. So $S_{21}(s)$ is analytic in the right half s-plane. Moreover,

$$\begin{aligned}|S_{21}(j\omega)|^2 = 4\frac{R_1}{R_2}\frac{|V_2(j\omega)|^2}{|E|^2} &= \frac{|V_2(j\omega)|^2}{2R_2}\frac{1}{\dfrac{1}{8}\dfrac{|E|^2}{R_1}} \\ &= \frac{\text{Power delivered to } R_2}{\text{Maximum power available from E}} \le 1\,,\end{aligned} \tag{8.26}$$

and $|S_{21}(j\omega)|^2$ is a measure of the fraction of the maximum power available from the voltage source that is transmitted to the load R_2. We call $|S_{21}(j\omega)|^2$ the *transmission power gain* of the two-port with respect to R_1 and R_2. The fact that it is less than one has implications in filter design and filter sensitivity, as we will see in the following chapters. To summarize, we have

Theorem 8.3. *The transmission function $S_{21}(s)$ has the following properties:*

1. *$S_{21}(s)$ is real when s is real;*
2. *$S_{21}(s)$ is analytic in the right-half s-plane;*
3. *$|S_{21}(s)| \le 1$ for all s such that $\Re(s) \ge 0$.*

A function which satisfies these conditions is called a *bounded real* function. In view of Theorem 8.1, the reflection functions $S_{11}(s)$ and $S_{22}(s)$ are bounded real.

8.1.4 Power considerations

With reference to Fig. 8.1, the total power delivered to the two-port is:

$$\begin{aligned}
P_{total} &= \frac{1}{2}\Re[V_1(j\omega)I_1^*(j\omega)] + \frac{1}{2}\Re[V_2(j\omega)I_2^*(j\omega)] &(8.27)\\
&= |a_1(j\omega)|^2 - |b_1(j\omega)|^2 + |a_2(j\omega)|^2 - |b_2(j\omega)|^2 &(8.28)\\
&= (a^*(j\omega))^T a(j\omega) - (b^*(j\omega))^T b(j\omega) &(8.29)\\
&= a^H(j\omega)[U - S^H(j\omega)S(j\omega)]a(j\omega) \geq 0 \quad \text{for all} \quad a(j\omega)\,, &(8.30)
\end{aligned}$$

where for any square matrix M of complex elements, $M^H = (M^*)^T$ is the *Hermitian* of M.

The last equation says that the bracketed matrix is positive semi-definite. Since all the scattering parameters are analytic in the right-half s plane, by analytic continuation, we have

Theorem 8.4. *The matrix* $[U - S^H(s)S(s)]$ *of an* RLC *two-port is positive semi-definite for* $\Re(s) \geq 0$.

Definition 8.1. A square matrix $M(s)$ whose elements are rational functions of a complex variable s is *bounded real* if:

1. Each element of $M(s)$ is real when s is real;
2. Each element of $M(s)$ is analytic in the right half s-plane;
3. $U - M^H(s)M(s)$ is positive semi-definite for $\Re(s) \geq 0$.

With this definition, we can state:

Theorem 8.5. *The scattering matrix of an* RLC *two-port is bounded real.*

8.1.5 Lossless two-port

If the two-port comprises inductors and capacitors, namely, it is lossless, the power delivered to it is identically zero for all values of the terminal voltages and currents. From Eq. (8.30) we have

$$a^H(j\omega)[U - S^H(j\omega)S(j\omega)]a(j\omega) \equiv 0 \quad \text{for all} \quad a(j\omega)\,. \tag{8.31}$$

To satisfy this condition, it is necessary and sufficient that

$$U - S^H(j\omega))S(j\omega) = 0\,. \tag{8.32}$$

A matrix $S(j\omega)$ that satisfies Eq. (8.32) is called a *unitary matrix* and we have:

Theorem 8.6. *The scattering matrix of a lossless two-port is unitary.*

Expanding Eq. (8.32), we obtain the following conditions on the scattering parameters:

$$|S_{11}(j\omega)|^2 + |S_{21}(j\omega)|^2 = 1\,; \tag{8.33}$$

$$S_{11}^*(j\omega)S_{12}(j\omega) + S_{21}^*(j\omega)S_{22}(j\omega) = 0\,; \tag{8.34}$$

$$|S_{12}(j\omega)|^2 + |S_{22}(j\omega)|^2 = 1\,; \tag{8.35}$$

$$|S_{11}(j\omega)|^2 = |S_{22}(j\omega)|^2\,. \tag{8.36}$$

The last relation is obtained if the two-port is reciprocal. In view of Eqs. (8.33) and (8.26), $|S_{11}(j\omega)|^2$ is a measure of the reflected power and we shall call it the *reflected power gain* with respect to R_1, and similarly for $|S_{22}(j\omega)|^2$ with respect to R_2. Eq. (8.33) is the key to synthesis of transfer functions. Given a transfer function $|H(j\omega)|^2$, we find $|S_{21}(j\omega)|^2$ by Eq. (8.25) and from Eq. (8.33) we get $|S_{11}(j\omega)|^2$ and then $S_{11}(s)$. Lastly we get the input impedance $Z_{in(1)}(s)$ from Eq. (8.24) and proceed to synthesize $Z_{in(1)}(s)$ as the impedance of a two-port terminated in R_2.

8.1.6 Examples

Example 8.1. Let us find the scattering parameters of the lossless two-port shown in Fig. 8.2. Solving for V_1 and V_2 in terms of E_1 and E_2, we get

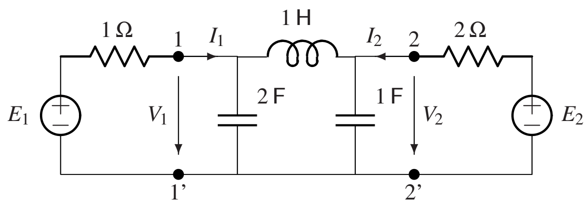

Fig. 8.2 A two-port used to illustrate the computation of scattering parameters

$$V_1(s) = \frac{(2s^2+s+2)E_1 + E_2}{4s^3+4s^2+7s+3}\,; \tag{8.37}$$

$$V_2(s) = \frac{2E_1 + (2s^2+s+1)E_2}{4s^3+4s^2+7s+3}\,. \tag{8.38}$$

Substituting into Eq. (8.18) - Eq. (8.20), we find

$$S_{11}(s) = \frac{-4s^3 - 5s + 1}{4s^3 + 4s^2 + 7s + 3}\,; \tag{8.39}$$

$$S_{12}(s) = \frac{2\sqrt{2}}{4s^3 + 4s^2 + 7s + 3}\,; \tag{8.40}$$

$$S_{21}(s) = \frac{2\sqrt{2}}{4s^3 + 4s^2 + 7s + 3}\,; \tag{8.41}$$

$$S_{22}(s) = \frac{-4s^3 - 5s - 1}{4s^3 + 4s^2 + 7s + 3}\,. \tag{8.42}$$

It is readily verified that

$$|S_{11}(j\omega)|^2 + |S_{21}(j\omega)|^2 = 1\,; \tag{8.43}$$

$$S_{11}(-j\omega)S_{12}(j\omega) + S_{21}(-j\omega)S_{22}(j\omega) = 0\,. \tag{8.44}$$

Example 8.2. A lossless two-port terminated in a one-ohm resistor at both ports is required to have a transfer function specified below. Find the impedance function across port-1 which is associated with this transfer function.

$$|H(j\omega)|^2 = \frac{1}{4}\,\frac{1}{1+\omega^6}\,. \tag{8.45}$$

By Eq. (8.78) the transmission power gain is

$$|S_{21}(j\omega)|^2 = 4\,\frac{R_1}{R_2}\,|H(j\omega)|^2 = \frac{1}{1+\omega^6}\,. \tag{8.46}$$

By Eq. (8.33), the reflection power gain is

$$|S_{11}(j\omega)|^2 = 1 - |S_{21}(j\omega)|^2 = \frac{\omega^6}{1+\omega^6}\,, \tag{8.47}$$

which by analytic continuation can be written as

$$S_{11}(s)S_{11}(-s) = \frac{-s^6}{1-s^6} = \frac{(s^3)(-s^3)}{(s^3+2s^2+2s+1)(-s^3+2s^2-2s+1)}\,. \tag{8.48}$$

We must assign all the left-hand plane poles to $S_{11}(s)$, whereas the factor s^3 may be assigned to either $S_{11}(s)$ or $S_{11}(-s)$. There are therefore two possible solutions:

$$S_{11,a}(s) = \frac{s^3}{s^3+2s^2+2s+1}\,, \tag{8.49}$$

$$S_{11,b}(s) = \frac{-s^3}{s^3+2s^2+2s+1}\,. \tag{8.50}$$

By Eq. (8.24), the two corresponding input impedances are:

$$Z_{in,a}(s) = \frac{2s^2+2s+1}{2s^3+2s^2+2s+1}, \tag{8.51}$$

$$Z_{in,b}(s) = \frac{2s^3+2s^2+2s+1}{2s^2+2s+1}. \tag{8.52}$$

One is the reciprocal of the other.

8.2 Scattering matrix with impedance terminations

So far, scattering matrix is defined with respect to resistive terminations. In practice, the load is often not a simple resistor but an *RLC* impedance. The signal source, likewise, usually has an internal impedance which is a combination of resistors, capacitors and inductors. This is the situation, for example, when a passive bandpass filter is inserted between a power amplifier and a transmitter antenna, or between a receiving antenna and a low noise amplifier.

To characterize a two-port with impedance terminations, we need to generalize the definition of scattering matrix. The generalization is not a simple matter of replacing R_1 with Z_1, etc. We wish to preserve the analytic and rational character of the scattering parameters, the unitary property, and the power relations among the parameters. At the same time, when we substitute R_1 for Z_1 in the generalization, we should revert to the case of resistive terminations.

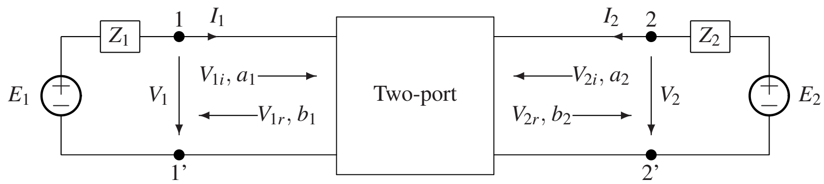

Fig. 8.3 A two-port with impedance terminations

8.2.1 Normalization factor

Consider the circuit of Fig. 8.3. The termination at port-1 is $Z_1(s)$ and at port-2 $Z_2(s)$. We recall from Sect. 8.1.1 that we normalize the incident and reflected voltages by the square root of the terminating resistance in order to obtain a definition of a_1 and b_1 such that $|a_1(j\omega)|^2$ and $|b_1(j\omega)|^2$ represent the incident power and reflected power, respectively, at port-1. In the general case, let $r_1(s^2)$ be an even function of s defined as

$$r_1(s^2) = \frac{1}{2}[Z_1(s) + Z_1(-s)] = h_1(s)h_1(-s) \tag{8.53}$$

It is factored into a product of rational functions $h_1(s)$ and $h_1(-s)$. We make the following observations about $r_1(s^2)$:

1. The poles and zeroes of $r_1(s^2)$ have quadrantal symmetry with respect to the real and imaginary axes in the complex plane, namely, they appear as $(s^2+\alpha s+\beta)(s^2-\alpha s+\beta)$ for complex poles and zeros and as $(-s^2+\alpha^2)$ for real ones.
2. $r_1(s^2)$ does not have any poles on the $j\omega$ axis because $Z_1(s)$ is a positive real function whose $j\omega$ poles are simple and are canceled by those of $Z_1(-s)$ when these poles are extracted by partial fraction expansion.
3. Zeros of $r_1(s^2)$ on the $j\omega$ axis are of even multiplicity because if $h_1(s)$ has a pair of zeros there, then $h_1(-s)$ must have them as well.
4. Because a pole of $Z_1(s)$ at $s=\infty$ will be canceled by that of $Z_1(-s)$, the degree of the numerator of $r_1(s^2)$ is not greater than that of the denominator.

Given $Z_1(s)$, there are many ways of assigning the factors of $r_1(s^2)$ to $h_1(s)$. Later, we will derive conditions on $h_1(s)$ and $h_1(-s)$ which will determine how the factors should be assigned.

In the same manner, at port-2, we have

$$r_2(s^2) = \frac{1}{2}[Z_2(s) + Z_2(-s)] = h_2(s)h_2(-s) \tag{8.54}$$

with the same properties as $r_1(s^2)$.

The rational functions $h_1(s)$ and $h_2(s)$ are called the *normalization factors* of port-1 and port-2, respectively.

8.2.2 Derivation

With reference to Fig. 8.3, let

$$V_1 = V_{1i} + V_{1r}\,, \tag{8.55}$$

where V_{1i} and V_{1r} are regarded as incident and reflected voltages defined as

$$V_{1i} = \frac{Z_1(-s)}{Z_1(s)+Z_1(-s)}[V_1 + Z_1(s)I_1] = \frac{Z_1(-s)}{2h_1(s)h_1(-s)}[V_1 + Z_1(s)I_1]\,, \tag{8.56}$$

$$V_{1r} = \frac{Z_1(s)}{Z_1(s)+Z_1(-s)}[V_1 - Z_1(-s)I_1] = \frac{Z_1(s)}{2h_1(s)h_1(-s)}[V_1 - Z_1(-s)I_1]\,. \tag{8.57}$$

As in Sect. 8.1.1, we define the incident and reflected wave quantities $a_1(s)$ and $b_1(s)$ as, with port-2 terminated in $Z_2(s)$:

$$a_1(s) = \frac{h_1(-s)}{\sqrt{2}Z_1(-s)} V_{1i} = \frac{1}{2\sqrt{2}h_1(s)} [V_1 + Z_1(s)I_1] , \tag{8.58}$$

$$b_1(s) = \frac{h_1(s)}{\sqrt{2}Z_1(s)} V_{1r} = \frac{1}{2\sqrt{2}h_1(-s)} [V_1 - Z_1(-s)I_1] . \tag{8.59}$$

We note that

$$|a_1(j\omega)|^2 = a_1(j\omega)a_1(-j\omega) = \frac{1}{2}\Re\left[\frac{1}{Z_1(j\omega)}\right] |V_{1i}(j\omega)|^2 , \tag{8.60}$$

$$|b_1(j\omega)|^2 = b_1(j\omega)b_1(-j\omega) = \frac{1}{2}\Re\left[\frac{1}{Z_1(j\omega)}\right] |V_{1r}(j\omega)|^2 , \tag{8.61}$$

and they can be interpreted as the incident and reflected power, respectively, as before.

For completeness, the corresponding quantities for port-2 are listed below.

$$V_2 = V_{2i} + V_{2r} \qquad \text{with} \tag{8.62}$$

$$V_{2i} = \frac{Z_2(-s)}{2h_2(s)h_2(-s)} [V_2 + Z_2(s)I_2] , \tag{8.63}$$

$$V_{2r} = \frac{Z_2(s)}{2h_2(s)h_2(-s)} [V_2 - Z_2(-s)I_2] . \tag{8.64}$$

With port-1 terminated in $Z_1(s)$ ($E_1 = 0$), we have

$$a_2(s) = \frac{1}{2\sqrt{2}h_2(s)} [V_2 + Z_2(s)I_2] , \tag{8.65}$$

$$b_2(s) = \frac{1}{2\sqrt{2}h_2(-s)} [V_2 - Z_2(-s)I_2] . \tag{8.66}$$

The port voltages and currents can be expressed in terms of the $a's$ and $b's$:

$$V_1 = \frac{\sqrt{2}Z_1(-s)}{h_1(-s)} a_1(s) + \frac{\sqrt{2}Z_1(s)}{h_1(s)} b_1(s) , \qquad I_1 = \frac{\sqrt{2}}{h_1(-s)} a_1(s) - \frac{\sqrt{2}}{h_1(s)} b_1(s) , \tag{8.67}$$

$$V_2 = \frac{\sqrt{2}Z_2(-s)}{h_2(-s)} a_2(s) + \frac{\sqrt{2}Z_2(s)}{h_2(s)} b_2(s) , \qquad I_2 = \frac{\sqrt{2}}{h_2(-s)} a_2(s) - \frac{\sqrt{2}}{h_2(s)} b_2(s) . \tag{8.68}$$

Finally the scattering matrix is defined as before:

$$\begin{bmatrix} b_1(s) \\ b_2(s) \end{bmatrix} = \begin{bmatrix} S_{11}(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} \begin{bmatrix} a_1(s) \\ a_2(s) \end{bmatrix} , \tag{8.69}$$

where the parameters are computed as follows:

$$S_{11}(s) = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \frac{h_1(s)}{h_1(-s)} \left.\frac{Z_{in(1)} - Z_1(-s)}{Z_{in(1)} + Z_1(s)}\right|_{E_2=0}; \tag{8.70}$$

$$S_{12}(s) = \left.\frac{b_1}{a_2}\right|_{a_1=0} = -2h_1(s)h_2(s)\left.\frac{I_1}{E_2}\right|_{E_1=0}; \tag{8.71}$$

$$S_{21}(s) = \left.\frac{b_2}{a_1}\right|_{a_2=0} = -2h_1(s)h_2(s)\left.\frac{I_2}{E_1}\right|_{E_2=0}; \tag{8.72}$$

$$S_{22}(s) = \left.\frac{b_2}{a_2}\right|_{a_1=0} = \frac{h_2(s)}{h_2(-s)} \left.\frac{Z_{in(2)} - Z_2(-s)}{Z_{in(2)} + Z_2(s)}\right|_{E_1=0}. \tag{8.73}$$

8.2.3 properties

We make the following observations about $S_{11}(s)$ and $S_{22}(s)$.

1. $S_{11}(s)$ is real when s is real since it is a rational function with real coefficients.
2. Later we will show that it is always possible to choose $h_1(s)$ and $h_2(s)$ such that

$$\frac{h_1(s)}{h_1(-s)} \quad \text{and} \quad \frac{h_2(s)}{h_2(-s)} \tag{8.74}$$

are analytic in the right half s-plane. Since $Z_{in(1)}(s)$ and $Z_1(s)$ are positive real functions, $S_{11}(s)$ is analytic in the right half s-plane.
3. Let $Z_{in(1)}(j\omega) = R_{in(1)}(\omega) + jX_{in(1)}(\omega)$ and $Z_1(j\omega) = R_1(\omega) + jX_1(\omega)$. Then $R_{in(1)}(\omega) \geq 0$ and $R_1(\omega) \geq 0$ for all ω. Moreover,

$$|S_{11}(j\omega)|^2 = \left|\frac{h_1(j\omega)}{h_1(-j\omega)}\right|^2 \left|\frac{Z_{in(1)}(j\omega) - Z_1(-j\omega)}{Z_{in(1)}(j\omega) + Z_1(j\omega)}\right|^2 \tag{8.75}$$

$$= \frac{\left[(R_{in(1)}(\omega) - R_1(\omega)\right]^2 + \left[X_{in(1)}(\omega) + X_1(\omega)\right]^2}{\left[(R_{in(1)}(\omega) + R_1(\omega)\right]^2 + \left[X_{in(1)}(\omega) + X_1(\omega)\right]^2} \leq 1. \tag{8.76}$$

It follows that

Theorem 8.7. *The scattering matrix parameters $S_{11}(s)$ and $S_{22}(s)$ are bounded real.*

Next we consider $S_{12}(s)$ and $S_{21}(s)$.

1. By reciprocity

$$\left.\frac{I_1}{E_2}\right|_{E_1=0} = \left.\frac{I_2}{E_1}\right|_{E_2=0}. \tag{8.77}$$

So, $S_{12}(s) = S_{21}(s)$.
2. $S_{21}(s)$ and the transfer function $H(s)$ are related:

$$S_{21}(s) = 2\frac{h_1(s)h_2(s)}{Z_2(s)}H(s). \tag{8.78}$$

3. $S_{21}(s)$ and $S_{12}(s)$ are real when s is real.
4. By Eq. (8.74), and since $H(s)$ has no poles in the right half s-plane, $S_{21}(s)$ is analytic in the right half s-plane.

   Lastly power consideration leads to the following properties.

1. The power taken by the two-port is

$$P(jw) = \frac{1}{2}\Re[V_1(j\omega)I_1(-j\omega)] + \frac{1}{2}\Re[V_2(j\omega)I_2(-j\omega)] \tag{8.79}$$

$$= |a_1(j\omega)|^2 + |a_2(j\omega)|^2 - |b_1(j\omega)|^2 - |b_2(j\omega)|^2 \tag{8.80}$$

$$= a^H(j\omega)\left[U - S^H(j\omega)S(j\omega)\right]a(j\omega) \geq 0 \quad \text{for all}\, a(j\omega)\,, \tag{8.81}$$

   for a passive two-port.
2. By analytic continuation, the matrix $[U - S^H(s)S(s)]$ is positive semi-definite for $\Re(s) \geq 0$.
3. If the two-port is lossless, $S(j\omega)$ is unitary.
4. If the two-port is lossless, $|S_{11}(j\omega)|^2 = |S_{22}(j\omega)|^2$.

These properties can be summarized in a theorem:

Theorem 8.8. *The scattering matrix of an* RLC *two-port with impedance terminations and with normalization as prescribed in Sect. 8.2.1 and Eq. (8.74), is bounded real.*

8.2.4 Assignment of normalization factors

In order to satisfy Eq. (8.74), $h_1(s)$ and $h_2(s)$ must not have any poles in the right half plane, and $h_1(-s)$ and $h_2(-s)$ must not have any zeros in the right half plane. This can always be accomplished since $r(s^2)$ is an even polynomial. As an example, let the terminating impedance be

$$Z(s) = \frac{s+3}{s+5}\,. \tag{8.82}$$

Then

$$r(s^2) = \frac{-s^2+15}{-s^2+25} = \frac{(s+\sqrt{15})(-s+\sqrt{15})}{(s+5)(-s+5)} = h(s)h(-s)\,. \tag{8.83}$$

The correct choice is

$$h(s) = \frac{-s+\sqrt{15}}{s+5}\,, \qquad h(-s) = \frac{s+\sqrt{15}}{-s+5}\,. \tag{8.84}$$

We shall call these rational functions the *principal* normalization factors of $h(s)$ and $h(-s)$, respectively. We note that $r(s^2)$ can also be expressed as

$$r(s^2) = \frac{(s+\sqrt{15})(-s+\sqrt{15})}{(s+5)(-s+5)} \frac{(s-a)}{(s+a)} \frac{(-s-a)}{(-s+a)}. \tag{8.85}$$

So

$$h(s) = \frac{-s+\sqrt{15}}{s+5} \frac{(s-a)}{(s+a)}, \qquad h(-s) = \frac{s+\sqrt{15}}{-s+5} \frac{(-s-a)}{(-s+a)}, \tag{8.86}$$

is also an acceptable choice. That is, we can multiply the principal normalization factor of $h(s)$ by any number of all-pass functions and $r(s^2)$ remains invariant. So the factorization of $r(s^2)$ is only unique up to the principal normalization factors. Unless otherwise stated, we shall assume that $r(s^2)$ is factored into principal normalization factors. The inclusion of all-pass functions is necessary in the synthesis of broadband matching circuits [14, 70, 17].

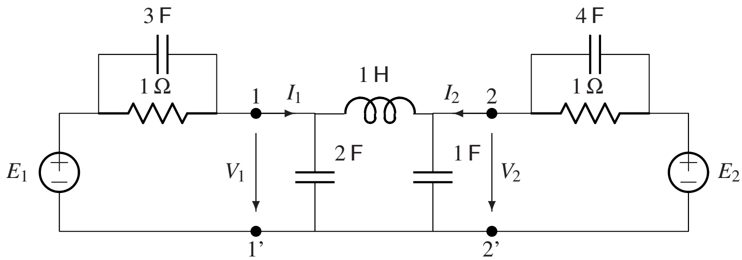

Fig. 8.4 Example of a lossless two-port terminated in impedances

Example 8.3. Figure 8.4 shows a lossless two-port terminated in an *RC* impedance at both ports. Let

$$Z_1(s) = \frac{1}{3s+1}, \qquad Z_2(s) = \frac{1}{4s+1}. \tag{8.87}$$

The normalization factors are found to be

$$h_1(s) = \frac{1}{3s+1}, \qquad h_2(s) = \frac{1}{4s+1}, \tag{8.88}$$

$$h_1(-s) = \frac{1}{-3s+1}, \qquad h_2(-s) = \frac{1}{-4s+1}. \tag{8.89}$$

Solving the circuit for V_1 and V_2, we get

$$V_1 = \frac{(5s^2+s+1)(3s+1)E_1 + (4s+1)E_2}{25s^3+10s^2+11s+2}; \tag{8.90}$$

$$V_2 = \frac{(3s+1)E_1 + (5s^2+s+1)(4s+1)E_2}{25s^3+10s^2+11s+2}. \tag{8.91}$$

The scattering parameters are

$$S_{11}(s) = \frac{-25s^3 - 9s}{25s^3 + 10s^2 + 11s + 2}, \tag{8.92}$$

$$S_{12}(s) = \frac{2}{25s^3 + 10s^2 + 11s + 2}, \tag{8.93}$$

$$S_{21}(s) = \frac{2}{25s^3 + 10s^2 + 11s + 2}, \tag{8.94}$$

$$S_{22}(s) = \frac{-25s^3 - 9s}{25s^3 + 10s^2 + 11s + 2}. \tag{8.95}$$

As a check, we find

$$|S_{11}(j\omega)|^2 = \frac{625\omega^6 - 450\omega^4 + 81\omega^2}{625\omega^6 - 450\omega^4 + 81\omega^2 + 4}, \tag{8.96}$$

$$|S_{21}(j\omega)|^2 = \frac{4}{625\omega^6 - 450\omega^4 + 81\omega^2 + 4}, \tag{8.97}$$

and their sum is one as it should be.

Remark 8.1. Scattering matrix has been a subject of study for a number of years. What we have presented here are fundamental properties just sufficient to support the synthesis of transfer functions. For a more comprehensive exposition, including realizability of scattering matrix and generalization to n-ports and to non-reciprocal circuits, see [15, 71, 72, 17].

Secondly, scattering matrix is often used to characterize a small-signal electronic circuit such as an amplifier operating at high frequencies. Instead of trying to find a detailed circuit model of the amplifier, we make measurements at its terminals and obtain a scattering matrix characterization of the active two-port, over a limited frequency range. Analysis and design of the circuit of which the amplifier is a part can now proceed in terms of the scattering parameters rather than the internal circuit elements, which are impossible to measure and difficult to make certain [22].

8.3 Gain-bandwidth limitations of two-ports

Let us remind ourselves that the transmission function $S_{21}(s)$ has to do with the power transmitted to a load. In fact, as shown in Eq. (8.26), $|S_{21}(j\omega)|^2$ is a measure of the transmitted power as a fraction of the maximum power available at the source. At the same time, $|S_{11}(j\omega)|^2 = 1 - |S_{21}(j\omega)|^2$ is a measure of the power not transmitted. In a well-designed filter as a lossless two-port terminated in a load, the transmission power gain $|S_{21}(j\omega)|^2$ will be close to unity in the passband and to zero in the stop-band. Conversely, the reflection power gain $|S_{11}(j\omega)|^2$ will be close to zero in the passband and to unity in the stop-band. In practice, the load invariably consists of a shunt capacitor in parallel with a resistor or with a more complex impedance. The presence of this shunt capacitor places limitations on how close $|S_{21}(j\omega)|^2$ can be to unity over the passband. The problem is then to find a bound on the transmission power gain and relate it to the shunt capacitance.

Bode studied this problem in the 1930s [8] in conjunction of the design of amplifiers. He asked if it was possible to insert a lossless two-port between an amplifier (vacuum tube) and an RC load to increase the useful bandwidth of the overall circuit? If it is, what are the tradeoffs between bandwidth and gain?

Consider the circuit shown in Fig. 8.5. Port-1 is terminated in a resistor and port-2 in an RC impedance $Z_2 = R/(1+sRC)$. We assume there is no internal capacitance across port-2. If there is, it can be absorbed in C. Let $Z_{in(2)}$ be the input impedance

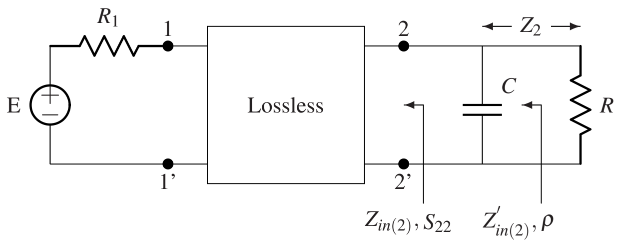

Fig. 8.5 A circuit whose gain-bandwidth limitation is under consideration

to the left of port-2 and let $S_{22}(s)$ be the reflection coefficient with respect to Z_2. Then from Eq. (8.73), we have

$$S_{22}(s) = \frac{h_2(s)}{h_2(-s)} \frac{Z_{in(2)} - Z_2(-s)}{Z_{in(2)} + Z_2(s)}, \tag{8.98}$$

for some suitable $h_2(s)$. Define $\rho(s)$ as the reflection coefficient with respect to R. Then we have

$$\rho(s) = \frac{Z'_{in(2)} - R}{Z'_{in(2)} + R}, \tag{8.99}$$

where $Z'_{in(2)}$ is the parallel combination of $Z_{in(2)}$ and $1/sC$. Combining equations, we find

$$|S_{22}(j\omega)| = |\rho(j\omega)|. \tag{8.100}$$

Now consider the function $1/\rho(s)$. It does not have any zeros in the right-half plane because $Z'_{in(2)}$ is positive real. However, it may have poles there. Let the poles in the right-half plane be $p_1, p_2, \ldots, p_n$. They are positive real or occur in complex conjugate pairs with positive real parts. We now multiply $1/\rho(s)$ with as many all-pass sections as there are poles in the right-half plane and construct the following function.

$$F(s) = \ln \frac{-1}{\rho(s)} = \ln \left[-\frac{Z'_{in(2)} + R}{Z'_{in(2)} - R} \times \frac{s - p_1}{s + p_1} \cdots \frac{s - p_n}{s + p_n} \right]. \tag{8.101}$$

We note that $F(s)$ is analytic in the right-half plane and on its boundary, and we have

$$\oint_{\mathscr{C}} F(s)\mathrm{d}s = 0\,, \tag{8.102}$$

where the contour $\mathscr{C}$ encloses the right-half plane and consists of the imaginary axis and the arc of a semi-circle of infinite radius. Expand the quantity inside the brackets in Eq. (8.101) about $s = \infty$ and we get:

$$\begin{aligned}\left.\frac{-1}{\rho(s)}\right|_{s\to\infty} &= \left(1+\frac{2}{sRC}\right)\left(1-\frac{2p_1}{s}\right)\cdots\left(1-\frac{2p_n}{s}\right) \\ &= \left[1+\frac{2}{s}\left(\frac{1}{RC}-\sum_k p_k\right)\right]\,.\end{aligned} \tag{8.103}$$

Substitute into Eq. (8.101) and integrate along $\mathscr{C}$. Noting that the real part is an even function of ω and the imaginary part is odd, and that $\ln(1+x) = x$ for small x, we get

$$\int_0^\infty \ln\frac{1}{|\rho(j\omega)|}\mathrm{d}\omega = \pi\left(\frac{1}{RC}-\sum_k p_k\right)\,. \tag{8.104}$$

Using Eq. (8.100) and noting the sum of p_k is positive, we obtain an integral bound on the reflection power gain $|S_{22}(j\omega)|$:

$$\int_0^\infty \ln\left|\frac{1}{S_{22}(j\omega)}\right|\mathrm{d}\omega \le \frac{\pi}{RC}\,. \tag{8.105}$$

We can state the result in a theorem.

Theorem 8.9. *In a lossless two-port terminated at the source end in a resistor and at the load end in a resistance R shunted by a capacitance C, the area under the curve of the inverse of the absolute value of the reflection coefficient at the input or output port is limited by π over the time constant RC.*

Assume the transmission power gain is constant at $|S_{21}|_o^2$ in the passband whose edges are ω_1 and ω_2. Invoking the unitary properties, we find the limitation on the constant transmission power gain to be as follows.

$$|S_{21}|_o^2 \le 1 - \mathrm{e}^{-2\pi/(\omega_2-\omega_1)RC}\,. \tag{8.106}$$

Following Bode, we define the product of bandwidth and time constant as

$$Q = (\omega_2 - \omega_2)RC\,, \tag{8.107}$$

and we have

$$|S_{21}|_o^2 \le 1 - \mathrm{e}^{-2\pi/Q}\,. \tag{8.108}$$

The larger the Q, the smaller the constant transmission power gain, meaning that for a fixed RC load, larger bandwidth can only be achieved at a price of lower gain. In Fig. 8.6, we have plotted Eq. (8.108) in solid line. The graph shows that with a value of Q less than one, the constant transmission power gain is nearly unity. As Q increases, e.g., by virtue of an increase in capacitance but at constant bandwidth, or vice versa, the constant transmission power gain drops slowly to zero.

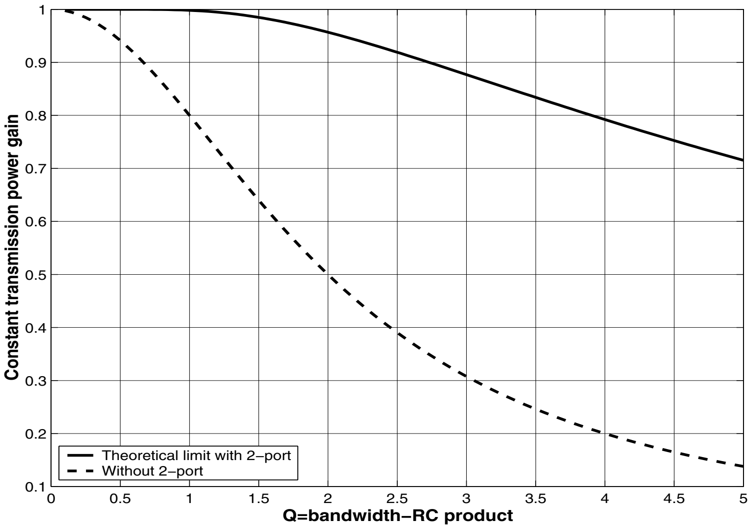

Fig. 8.6 Theoretical limit on the constant transmission power gain

We have also plotted, in dashed line, the transmission power gain for the case without the two-port, namely, where the RC load is connected directly to the source with internal resistance R_1, assumed to be equal to R for simplicity. It is seen that the solid line is always above the dashed line, implying that it is theoretically possible to find a two-port inserted between a load and a source such that the bandwidth over which the transmission power gain $|S_{21}|_o^2$ is constant is larger than that of a circuit without the two-port.

8.3.1 Examples - gain-bandwidth tradeoffs

We will illustrate the concept of gain-bandwidth tradeoffs with two examples. The first is to show how we can insert a lossless two-port between a parallel RC load and a source to increase the useful bandwidth. The second is to show how we can trade bandwidth for gain over a narrow band in a low-noise amplifier circuit.

Example 8.4. Consider the circuit of Fig. 8.7(a), in which the RC load may represent the input circuit of an amplifier. If it is connected directly to a source, the transmission power gain is:[1]

$$|S_{21}(j\omega)|^2 = \frac{4}{\omega^2+4}\,. \tag{8.109}$$

It is plotted in dash line in Fig. 8.8.

Let us insert an *LC* two-port between the load and source, as shown in Fig. 8.7**b**. The termination at port-2 is an impedance $Z_2(s) = 1/(s+1)$. By Eq. (8.54), we have

$$r_2 = \frac{1}{2}\left[Z_2(s)+Z_2(-s)\right] = \frac{1}{(s+1)(-s+1)} = h_2(s)h_2(-s)\,, \tag{8.110}$$

$$h_2(s) = \frac{1}{s+1}\,. \tag{8.111}$$

The termination at port-1 is a resistor and $h_1(s) = 1$. By Eq. (8.78), we obtain the transmission function and transmission power gain given below.

$$S_{21}(s) = 2\frac{h_1(s)h_2(s)}{Z_2(s)}H(s) = \frac{2}{s^3+2s^2+3s+2}\,; \tag{8.112}$$

$$|S_{21}(j\omega)|^2 = \frac{4}{\omega^6-2\omega^4+\omega^2+4}\,. \tag{8.113}$$

Plotted in solid line in Fig. 8.8, the transmission power gain shows a marked increase in the constant-gain bandwidth while the half-power bandwidth is reduced. In other words, the inserted two-port broadens the range of frequency over which maximum power is transferred to the load. It serves the function of matching the complex load to the source resistor over a broader band as compared to the band without the two-port. For this reason, the two-port is often called a *matching two-port.*

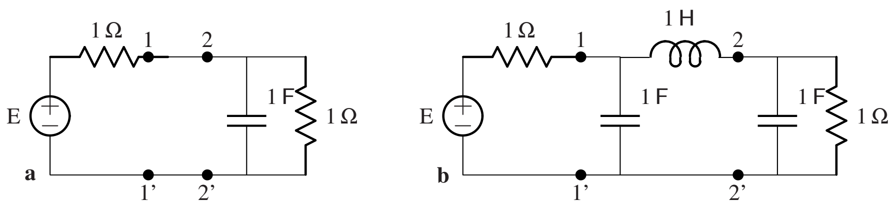

Fig. 8.7 **a** A circuit in which the load is connected directly to the source; **b** Insertion of a two-port to achieve constant gain over a broader band but reduced half-power bandwidth

Example 8.5. The input circuit of a radio-frequency low-noise amplifier [22], after normalization and much simplification, consists of a series connection of a capacitor and a resistor, as shown in Fig. 8.9. We insert a simple *LC* two-port between the

[1] We have normalized all the element values with respect to the principal frequency of interest and to the source resistance, for the sake of brevity and clarity.

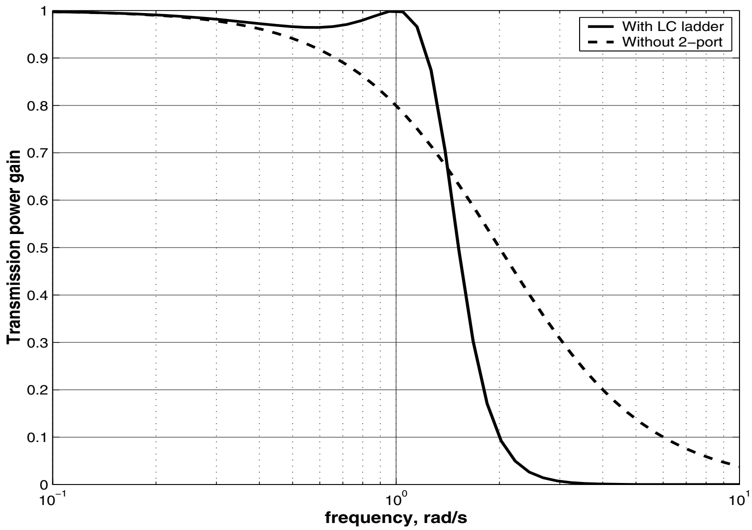

Fig. 8.8 Transmission power gain of a circuit with and without a broad-banding two-port inserted between the source and load

source and load to effect maximum power transfer over a narrow band (e.g., 869 – 894 MHz at a base station in a North America digital cellular system [35]). Let us compare the transmission power gain of the circuit with the two-port with that without the two-port. Following the same steps as Example 8.4, we have $Z_2(s) =$

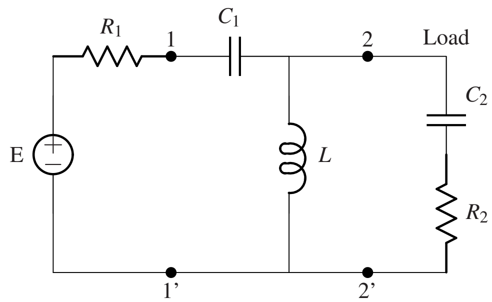

Fig. 8.9 Insertion of a two-port in the input circuit of a low-noise amplifier to increase the transmission power gain over a narrow band

$10+10/s$, $h_2(s) = \sqrt{10}$, and $h_1(s) = 1$. The transmission function is found to be:

$$S_{21}(s) = \frac{2\sqrt{R_1R_2}LC_1C_2s^3}{s^3LC_1C_2(R_1+R_2)+s^2(LC_1+LC_2+R_1R_2C_1C_2)+s(R_1C_1+R_2C_2)+1}\,. \tag{8.114}$$

If the load is connected directly to the source, the transmission function is:

$$S_{21,direct}(s) = \frac{2\sqrt{R_1 R_2} s C_2}{s C_2 (R_1 + R_2) + 1}. \tag{8.115}$$

The two transmission power gains are plotted in Fig. 8.10, for the following element values: $R_1 = 1\,\Omega$, $R_2 = 10\,\Omega$, $C_1 = 0.2294\,\mathsf{F}$, $C_2 = 0.1\,\mathsf{F}$, and $L = 3.7321\,\mathsf{H}$. It is evident that the insertion of the matching two-port results in maximum power transfer over a narrow band, which is much desired in a low-noise amplifier.

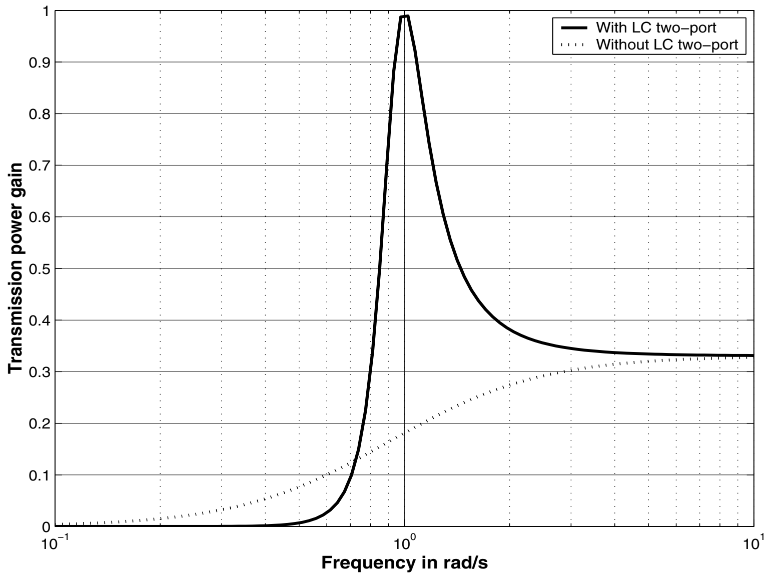

Fig. 8.10 Transmission power gain of the input circuit of a low-noise amplifier with and without a matching two-port

8.3.2 Remarks

The gain-bandwidth limitations are valid for small-signal electronic circuits as well. Indeed, the study of these limits was motivated by broadband amplifier design. The source could be a power amplifier and the load an antenna, or the source could be an antenna and the load a low noise amplifier. In either case, the shunt capacitance plays a significant role in determining the gain-bandwidth limitations. Since the publication of Bode's work, there had been a number of important contributions to the theory of broadband circuit design [23, 24, 70, 15, 72]. The later works generalized the gain-bandwidth limitations to the case of arbitrary terminating impedance. In practice, the actual design can alternatively be accomplished by computer-aided optimization (see Chapter 11). Meanwhile we should note that in modern radio-frequency integrated circuits operating in a narrow band at GHz the matching two-ports are very simple, usually consisting of an inductance and a capacitance of such values as to effect maximum power transfer at a single frequency [22, 41]. At this frequency, the reflection coefficient at port-1 $S_{11}(s) = 0$, namely the input impedance across port-1 is equal to the source resistance R_1, and maximum power is transferred to the load. See Problem 8.10.

8.4 Open-circuit impedance matrix

There are other characterizations of a two-port besides scattering matrix. These are strictly circuit parameters derived entirely in terms of voltages and currents at the ports, independent of their terminations.

8.4.1 Definition

With reference to Fig. 8.1, the terminal voltages V_1 and V_2 are linearly related to the terminal currents I_1 and I_2:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} . \tag{8.116}$$

The matrix $[z(s)] = [z_{ij}(s)]$ is called the *open-circuit impedance matrix*. Its coefficients or parameters are defined as:

$$z_{11}(s) = \left.\frac{V_1}{I_1}\right|_{I_2=0} , \tag{8.117}$$

$$z_{12}(s) = \left.\frac{V_1}{I_2}\right|_{I_1=0} , \tag{8.118}$$

$$z_{21}(s) = \left.\frac{V_2}{I_1}\right|_{I_2=0} , \tag{8.119}$$

$$z_{22}(s) = \left.\frac{V_2}{I_2}\right|_{I_1=0} . \tag{8.120}$$

In words, $z_{11}(s)$ is the input impedance at port-1 when port-2 is open-circuited ($R_2 = \infty$), and $z_{12}(s)$ is the voltage/current transfer function at port 1 when a current I_2 is applied at port 2 and port 1 is open-circuited ($R_1 = \infty$). Similarly for the other two. These coefficients are called *open-circuit impedance parameters*. The simplest way to compute them is to apply a current source across each of the two ports and solve for the terminal voltages.

The open-circuit impedance parameters have the following properties:

1. They are analytic in the right-half s plane, being all circuit functions;
2. They have the same poles, except for possible cancelation;
3. $z_{11}(s)$ and $z_{22}(s)$ are positive real functions, being input impedances;
4. $z_{12}(s) = z_{21}(s)$, by reciprocity.

Moreover, let port-1 be terminated in R_1 and port-2 in R_2 as in Fig. 8.1. Then the voltage transfer function $H(s) = V_2/E$ is

$$H(s) = \left.\frac{V_2}{E_1}\right|_{E_2=0} = \frac{z_{21}(s)R_2}{(R_1 + z_{11}(s))(R_2 + z_{22}(s)) - z_{12}(s)z_{21}(s)}. \tag{8.121}$$

So the zeros of $z_{21}(s)$ are the zeros of the transfer function.

Example 8.6. Consider the lossless two-port of Fig. 8.2. Following the suggestion of a simple way to compute the open-circuit parameters, we find them to be:

$$z_{11}(s) = \frac{s^2+1}{2s^3+3s} = \frac{\frac{1}{3}}{s} + \frac{\frac{1}{3}s}{2s^2+3}; \tag{8.122}$$

$$z_{12}(s) = \frac{1}{2s^3+3s} = \frac{\frac{1}{3}}{s} + \frac{-\frac{2}{3}s}{2s^2+3}; \tag{8.123}$$

$$z_{21}(s) = z_{12}(s); \tag{8.124}$$

$$z_{11}(s) = \frac{2s^2+1}{2s^3+3s} = \frac{\frac{1}{3}}{s} + \frac{\frac{4}{3}s}{2s^2+3}. \tag{8.125}$$

We see $z_{11}(s)$ and $z_{22}(s)$ are LC impedance functions and at the zeros of $z_{21}(s)$ ($s = \infty$), the output voltage is zero.

8.4.2 *Positive real matrix*

We recall that the impedance function of a circuit is a positive real function. We extend this concept to the open-circuit impedance matrix.

Definition 8.2. A matrix $M(s)$ is called a positive real matrix if and only if

1. Each element of $M(s)$ is real when s is real;
2. Each element of $M(s)$ is analytic in the right-half s plane;
3. $[M(s) + M^H(s)]$ is positive semi-definite for $\Re(s) \geq 0$.

Now consider the interior of the two-port of Fig. 8.1. Let R_k be the resistance of the kth resistor, L_k the inductance of the kth inductor, and C_k the capacitance of the kth capacitor. Let I_{R_k} and V_{R_k} be the current and voltage of R_k, and similarly for I_{L_k}, V_{L_k}, I_{C_k} and V_{C_k}. By Tellegan's theorem, we have

$$V_1 I_1^* + V_2 I_2^* = [I_1^* \quad I_2^*] \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix},$$
$$= \sum_k R_k I_{R_k} I_{R_k}^* + s \sum_k L_k I_{L_k} I_{L_k}^* + \frac{1}{s} \sum_k \frac{1}{C_k} I_{C_k} I_{C_k}^* \,. \tag{8.126}$$

Adding $V_1^* I_1 + V_2^* I_2$ to the above, we obtain

$$\frac{1}{2}[I_1^* \quad I_2^*][z(s) + z^H(s)] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$
$$= \sum_k R_k I_{R_k} I_{R_k}^* + \sigma \sum_k L_k I_{L_k} I_{L_k}^* + \frac{\sigma}{\sigma^2 + \omega^2} \sum_k \frac{1}{C_k} I_{C_k} I_{C_k}^*$$
$$\geq 0 \quad \text{for} \quad \sigma \geq 0 \,. \tag{8.127}$$

It follows that

Theorem 8.10. *The open-circuit impedance matrix is a positive real matrix.*

As a consequence, noting $[z(s)]$ is symmetric, we have, similar to the necessary and sufficient conditions of a positive real function,

Theorem 8.11. *A rational impedance matrix $[z(s)]$ is positive real if and only if*

1. *Each element of $[z(s)]$ is real when s is real;*
2. *Each element is analytic in the right-half s plane;*
3. *Poles on the $j\omega$ axis are simple and the matrix of residues at each pole is a non-negative definite matrix;*
4. *The matrix $\Re[z(j\omega)]$ is non-negative definite for all ω.*

We will omit the proof. The implication of item 3 is that the residues at each pole on the $j\omega$ axis satisfies the inequalities below, known as the *residue condition*:

$$k_{11} \geq 0 \,, \qquad k_{22} \geq 0 \,, \qquad k_{11}k_{22} - k_{12}k_{21} \geq 0 \,, \tag{8.128}$$

where k_{ij} is the residue of $z_{ij}(s)$ at a pole on the $j\omega$ axis.

8.4.3 Lossless two-port

If the two-port is lossless, the resistive term in Eq. (8.126) is zero and the right-hand side is an odd function of s. Because each element of $[z(s)]$ is analytic in the right-half s plane, we have the following theorem.

Theorem 8.12. *For a lossless two-port, the open-circuit impedance matrix has the following properties:*

1. *Each element is an odd function of s;*
2. *All the poles are on the $j\omega$ axis;*

3. *$z_{11}(s)$ and $z_{22}(s)$ are* LC *impedance functions;*
4. *$[z(s)]$ can be expanded as:*

$$\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} = \frac{1}{s}\begin{bmatrix} k_{0(11)} & k_{0(12)} \\ k_{0(21)} & k_{0(22)} \end{bmatrix} + s\begin{bmatrix} k_{\infty(11)} & k_{\infty(12)} \\ k_{\infty(21)} & k_{\infty(22)} \end{bmatrix} + \sum_i \frac{2s}{s^2+\omega_i^2}\begin{bmatrix} k_{i(11)} & k_{i(12)} \\ k_{i(21)} & k_{i(22)} \end{bmatrix}, \tag{8.129}$$

with

$$k_{p(11)} \geq 0 \qquad k_{p(22)} \geq 0 \qquad k_{p(11)}k_{p(22)} - k_{p(12)}k_{p(21)} \geq 0\,. \tag{8.130}$$

It is possible that for some p, $k_{p(11)} = 0$, $k_{p(12)} = 0$ and $k_{p(22)} \neq 0$. This situation occurs when there is an impedance in series with port-2. When port-2 is open-circuited, this impedance is not included in $z_{p(11)}$ or $z_{p(12)}$. We call the poles associated with the series impedance at port-2 the *private poles* of $z_{p(22)}$. Similarly, the private poles of z_{11} are those associated with impedances in series with port-1.

If for some ω_p, $k_{p(11)}k_{p(22)} - k^2_{p(12)} = 0$, we call the poles $s = \pm j\omega_p$ *compact poles*. In the example, all the poles are compact.

8.4.4 Scattering matrix and open-circuit impedance matrix

In practice, it is difficult to create an open-circuit, especially at high frequencies. To obtain a characterization of a two-port in terms of its open-circuit impedance matrix, it is more convenient to obtain the scattering matrix first and convert it to the impedance matrix. The conversion formulas for resistive termination and for impedance termination are, respectively:

$$[z(s)] = \sqrt{R}[U + S(s)][U - S(s)]^{-1}\sqrt{R}; \tag{8.131}$$

$$[z(s)] = [h^{-1}(-s)Z(-s) + h^{-1}(s)Z(s)S(s)][h^{-1}(-s) - h^{-1}(s)S(s)]^{-1}, \tag{8.132}$$

where

$$\sqrt{R} = \begin{bmatrix} \sqrt{R_1} & 0 \\ 0 & \sqrt{R_2} \end{bmatrix}, \quad Z(s) = \begin{bmatrix} Z_1(s) & 0 \\ 0 & Z_2(s) \end{bmatrix}, \quad h(s) = \begin{bmatrix} h_1(s) & 0 \\ 0 & h_2(s) \end{bmatrix}. \tag{8.133}$$

8.5 Short-circuit admittance matrix

We can also characterize a two-port in terms of admittance. With reference to Fig. 8.1, expressing the terminal currents in terms of the terminal voltages, we get

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \tag{8.134}$$

where the matrix $[y(s)] = [y_{ij}(s)]$ is called the *short-circuit admittance matrix*. Its coefficients are defined as follows.

$$y_{11}(s) = \left.\frac{I_1}{V_1}\right|_{V_2=0}; \tag{8.135}$$

$$y_{12}(s) = \left.\frac{I_1}{V_2}\right|_{V_1=0}; \tag{8.136}$$

$$y_{21}(s) = \left.\frac{I_2}{V_1}\right|_{V_2=0}; \tag{8.137}$$

$$y_{22}(s) = \left.\frac{I_2}{V_2}\right|_{V_1=0}. \tag{8.138}$$

So they are determined with either port terminated in a short-circuit. Obviously the short-circuit admittance matrix $[y(s)]$ and the open-circuit impedance matrix $[z(s)]$ are related:

$$[y(s)] = [z(s)]^{-1} \tag{8.139}$$

and they have the same analytic properties, namely, $y(s)$ is a positive real matrix. In addition, any admittance in parallel with port-2 does not appear in y_{11} and the associated poles are the private poles of y_{22}. Similarly for port-1. As an example, the short-circuit admittance parameters of the lossless two-port of Fig. 8.2 are:

$$y_{11}(s) = C_1 s + \frac{1}{sL} = 2s + \frac{1}{s}, \tag{8.140}$$

$$y_{12}(s) = y_{21}(s) = -\frac{1}{sL} = -\frac{1}{s}, \tag{8.141}$$

$$y_{22}(s) = C_2 s + \frac{1}{sL} = s + \frac{1}{s}. \tag{8.142}$$

The term $C_1 s$ is a private pole of $y_{11}(s)$, and $C_2 s$ is a private pole of $y_{22}(s)$.

Problems

8.1. Find the scattering matrix for each of the two-ports in Fig. 8.11. Assume port-1 is terminated in R_1 and port-2 in R_2.

8.2. The voltage transfer function of a lossless two-port terminated in R_1 at port-1 and R_2 at port-2 has the following low-pass characteristics. What is the largest allowable value of K?

$$|H(j\omega)|^2 = \frac{K^2}{1+\omega^{2n}}. \tag{8.143}$$

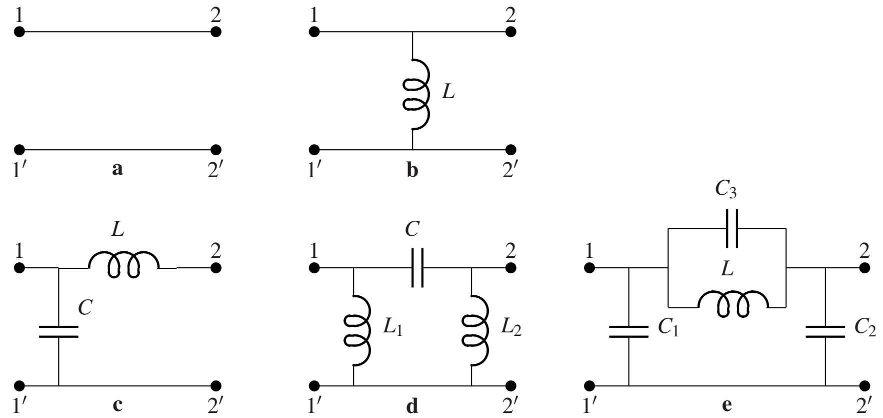

Fig. 8.11 Examples of two-ports for Problem 8.1

8.3. The voltage transfer function of a lossless two-port terminated in a resistance of 1 Ω at both ports is given below. Find the input impedance function at port-1. Is this impedance function unique?

$$|H(j\omega)|^2 = \frac{\omega^4+2\omega^2+1}{9\omega^6-26\omega^4+33\omega^2+4}. \tag{8.144}$$

8.4. The transmission power gain $|S_{21}(j\omega)|^2$ of a lossless two-port which is terminated at port-1 in 2 Ω and at port-2 in 1 Ω is given below. Find an input impedance function associated with it at port-1.

$$|S_{21}(j\omega)|^2 = \frac{8}{16\omega^6-12\omega^4+9}. \tag{8.145}$$

8.5. Find the scattering matrix of the lossless two-port terminated in resistances shown in Fig. 8.12. Verify that your answer is correct by showing that the parameters satisfy the unitary properties.

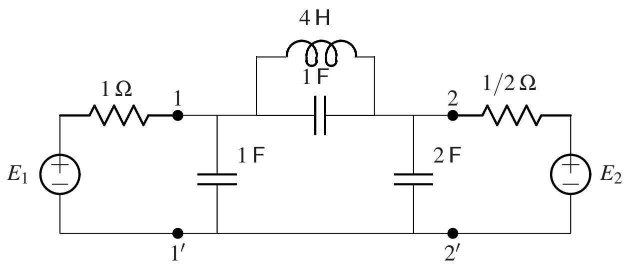

Fig. 8.12 A lossless two-port with resistive terminations

8.6. Find the scattering matrix of the lossless two-port terminated in *RC* impedances given in Fig. 8.13. Verify that your answer is correct by showing that the parameters satisfy the unitary properties.

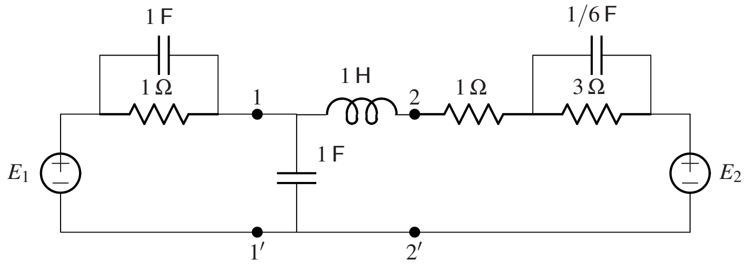

Fig. 8.13 A lossless two-port with impedance terminations

8.7. In Fig. 8.5, let $R_1 = R = 50\,\Omega$ and $C = 10\,\mathrm{pF}$. What is the limit on the constant transmission power gain if the passband width is 10 MHz? What is the answer if the capacitance is reduced to 1 pF?

8.8. In the circuit of Fig. 8.7**b**, insert an inductance between the source and load and see if the transmission power gain is improved in bandwidth. Experiment with different values of the inductance.

8.9. For the circuit of Fig. 8.14, derive a bound on the transmission power gain in terms of the capacitance C and resistance R_1 .

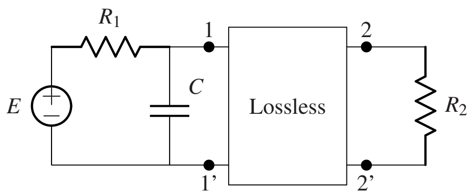

Fig. 8.14 The presence of capacitance at the input of a two-port places a limit on the constant transmission power gain

8.10. With reference to Example 8.5, the load represents the input equivalent circuit of the first transistor of a low-noise amplifier [22] and the source the equivalent circuit of an antenna, much simplified. The function of the matching two-port is to effect maximum power transfer from the source to the load. In practice, the element values of the source resistance R_1, load resistance R_2 and capacitance C_2 are known. The problem is to find the value of the inductance L and capacitance C_1 such that the input impedance across port-1 equals R_1 at some specified frequency. Under this condition, the transmission power gain is one and the load receives maximum power. Let $R_1 = 50\,\Omega$, $R_2 = 500\,\Omega$, $C_2 = 0.3611$ pF, and the frequency of interest $f_o = 881.5$ MHz. Find L and C_1 such that the transmission power gain $|S_{21}(j2\pi f_o)|^2 = 1$. Plot the response and compare it with that of the example. Is the bandwidth sufficient to cover the transmit band of the base cell station?

8.11. The output circuit of a low-noise amplifier [22], after simplification and normalization of the element values, is shown in Fig. 8.15. The source impedance is a parallel RC combination and the load is simply a resistor. A matching two-port is inserted between the source and load. Find the value of the inductance L and capacitance C_2 such that maximum power is transferred from the source to the load at frequency 1 rad/s. The source resistance $R_1 = 10\,\Omega$, and capacitance $C_1 = 0.1$ F, and the load resistance $R_2 = 1\,\Omega$. The transmission power gain should be as shown in Fig. 8.16.

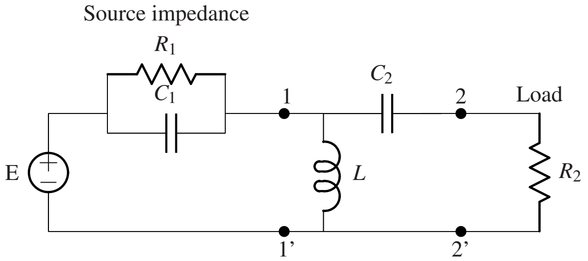

Fig. 8.15 The output circuit of a low-noise amplifier. The lossless two-port inserted between the source and load leads to maximum power transfer to the load.

8.12. Repeat Problem 8.11 for a more realistic set of element values: $R_1 = 500\,\Omega$, $R_2 = 50\,\Omega$, $C_1 = 0.13263$ pF, and the frequency at which the transmission power gain is one is $f_o = 2.4$ GHz. (Answer: $C_2 = 0.4421$ pF, $L = 8.2893$ nH.)

8.13. Find the open-circuit impedance matrix and short-circuit admittance matrix for each of the two-ports of Problem 8.1 if they exist. Demonstrate that the residue condition is satisfied in each case. Identify the private poles.

8.14. Obtain the open-circuit impedance matrix of the two-port of Problem 8.5. Identify the zeros of z_{12}. Are they the same as those of S_{12}?

8.15. Derive the properties of the open-circuit impedance matrix of an RC two-port. In particular, what are the residue conditions?

8.16. Consider a small-signal MOSFET. Assume the gate current is identically zero and the drain to source circuit is simply a transconductance whose current is controlled by the gate to source voltage. Let port-1 be the gate to source port and port-2 be the gate to source port. Let the two-port be terminated in resistors. Find the scattering matrix of the two-port which is the small-signal MOSFET.

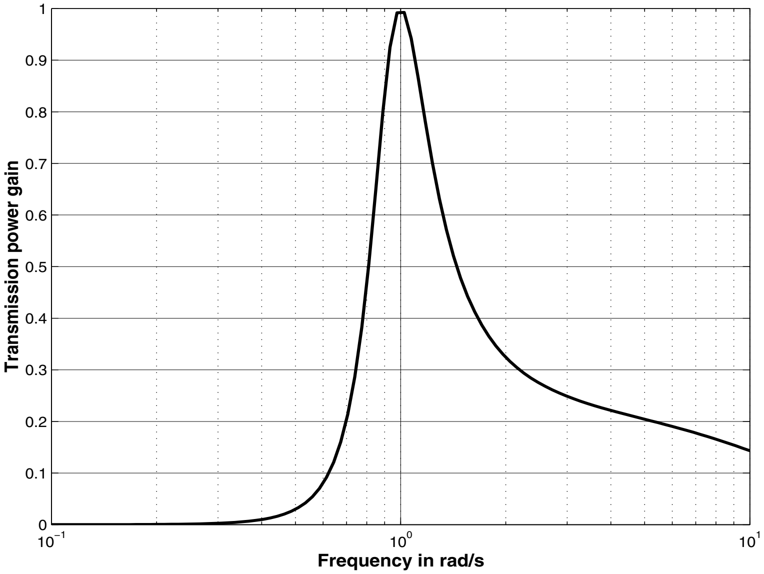

Fig. 8.16 The transmission power gain of the output circuit of a low-noise amplifier, showing maximum power transfer is possible over a narrow band with the insertion of a matching two-port

8.17. In practice, it is often convenient to determine the scattering matrix of a two-port, e.g., a transistor, experimentally with standard resistive terminations. When the two-port is inserted between an actual load impedance and a voltage source with internal impedance, the scattering matrix has to be "re-normalized" before we can proceed with analysis and design. Let $S_R(s)$ be the scattering matrix with resistive terminations and let $S_Z(s)$ be the scattering matrix with impedance terminations. How does one get $S_Z(s)$ from $S_R(s)$?

8.18. A length of transmission line may be considered as a two-port insofar as the terminal behavior is concerned. Consider a line with per unit inductance ℓ H/m, per unit resistance $r\,\Omega$ /m, per unit capacitance c F/m, and per unit conductance g S/m. The line is characterized in the frequency domain by what are known as the telegraphist's equations:

$$\frac{\mathrm{d}V(x,s)}{\mathrm{d}x} = -Z(s)I(x,s)\,, \tag{8.146}$$

$$\frac{\mathrm{d}I(x,s)}{\mathrm{d}x} = -Y(s)V(x,s)\,, \tag{8.147}$$

where $V(x,s)$ is the voltage at point x along the line, $I(x,s)$ is the current at point x in the direction of $+x$, $Z(s) = r + s\ell$, and $Y(s) = g + sc$. Let port-1 be at $x = 0$ and port-2 be at $x = d$, d being the length of the line. Show that the two-port is characterized by the following equations:

$$\begin{bmatrix} V(0,s) \\ I(0,s) \end{bmatrix} = \begin{bmatrix} \cosh\theta d & Z_c \sinh\theta d \\ \dfrac{1}{Z_c}\sinh\theta d & \cosh\theta d \end{bmatrix} \begin{bmatrix} V(d,x) \\ I(d,x) \end{bmatrix}, \tag{8.148}$$

where

$$Z_c(s) = \sqrt{\frac{r+s\ell}{g+sc}} \tag{8.149}$$

is the *characteristic impedance* of the line, and

$$\theta = \sqrt{(r+s\ell)(g+sc)} \tag{8.150}$$

is the *propagation constant*. The matrix that relates the input quantities to the output quantities is called the *ABCD* or *transmission* matrix denoted by T, defined as follows.

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cosh\theta d & Z_c \sinh\theta d \\ \dfrac{1}{Z_c}\sinh\theta d & \cosh\theta d \end{bmatrix}. \tag{8.151}$$

The transmission matrix is most useful in analysis of transmission line circuits consisting of a cascade connection of line segments. If T_1 and T_2 are the transmission matrices of two line segments, the overall transmission matrix is the product of the two:

$$T = T_1 T_2 \tag{8.152}$$

8.19. With reference to Problem 8.18, let port-1 be terminated in R_1 and port-2 in R_2. Show that the scattering matrix of the two-port which is the transmission line is:

$$S(s) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{8.153}$$

where

$$S_{11} = S_{22} = \frac{A + B/R_2 - R_1 C - R_1 D/R_2}{A + B/R_2 + R_1 C + R_1 D/R_2}, \tag{8.154}$$

$$S_{12} = S_{21} = 2\sqrt{\frac{R_1}{R_2}}\,\frac{1}{A + B/R_2 + R_1 C + R_1 D/R_2}. \tag{8.155}$$

8.20. With reference to Problem 8.18, let port-2 be terminated in an impedance Z_L. Show that the input impedance at port-1 is given by

$$Z_{in} = \frac{AZ_L + B}{CZ_L + D} = Z_c \frac{Z_L \cosh\theta d + Z_c \sinh\theta d}{Z_L \sinh\theta d + Z_c \cosh\theta d} \tag{8.156}$$

Thus the characteristic impedance Z_c of a line is the input impedance at port-1 when port-2 is terminated in Z_c. Show that $Z_c(s)$ is a positive real function under the extended definition.

8.21. Let us repeat Pupin's experiment on insertion of loading coils to increase the effective transmission length of a lossy transmission line. Consider four segments of transmission line connected in cascade, each 1 meter long. Let the line parameters be: $r = 0.25\,\Omega$ /m, $\ell = 0.005\,\mathrm{H/m}$, $c = 0.3\,\mathrm{F/m}$, and $g = 0\,\mathrm{S/m}$. Let port-1 be terminated in $R_1 = 1\,\Omega$ and port-2 in $R_2 = 1\,\Omega$. Compute and plot the transmission power gain $|S_{21}(j\omega)|^2$ of the circuit over some suitable frequency range. Now between each pair of lines, insert a "loading coil" of inductance $0.63\,\mathrm{H}$. Compute and plot the frequency response. Fig. 8.17 shows the frequency response with and without the loading coils. Notice how the loading coils have broadened the frequency response, thus allowing the signal to reach farther on the line. Experiment with different values of the loading coil inductance and see how it affects the bandwidth. Vary the line resistance and adjust the loading coil inductance to increase the bandwidth. What does Bode's gain-bandwidth limitation have to say about your results?

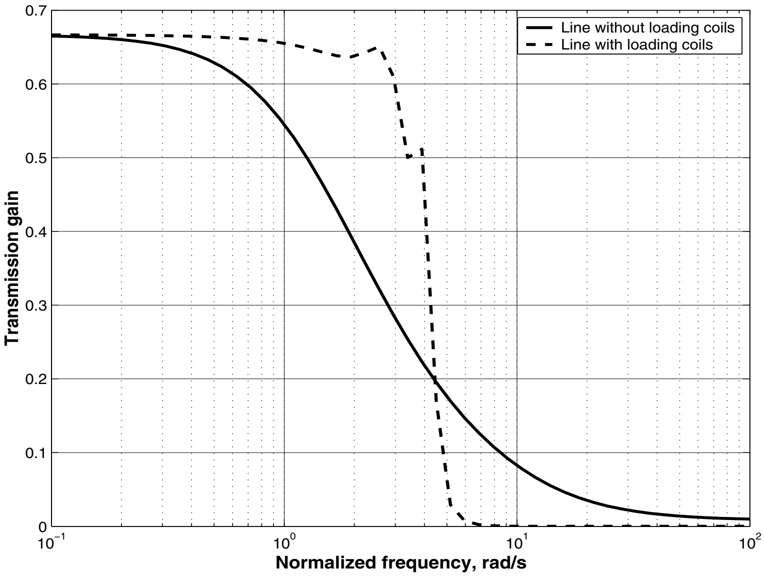

Fig. 8.17 Transmission gain of four segments of lossy transmission line with and without periodically inserted loading coils

Chapter 9
Synthesis of Transfer Functions

The most important problem in circuit design is the synthesis of transfer functions. The problem is to find a two-port terminated in known resistors such that its transfer function has a desired frequency response. The frequency response is often specified as a graphical sketch of magnitude or group delay versus frequency. We usually approach the problem in two steps. The first is to obtain a rational function of s whose frequency response approximates the specified frequency response. This is an approximation problem which we will take up in the next chapter. The second step is to synthesize the rational function we have just obtained as the transfer function of a two-port terminated in resistances, and it is the subject of this chapter.

9.1 The synthesis problem

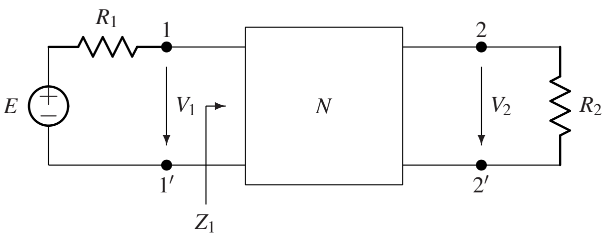

Fig. 9.1 A two-port terminated in R_1 and R_2 at port-1 and port-2, respectively

With reference to Fig. 9.1, suppose the magnitude squared of the transfer function $H(s)$ is specified as

$$|H(j\omega)|^2 = \left|\frac{V_2(j\omega)}{E}\right|^2 = \frac{a_m\omega^{2m} + \cdots + a_1\omega^2 + a_0}{b_n\omega^{2n} + \cdots + b_1\omega^2 + b_0}. \tag{9.1}$$

We wish to find a two-port such that its transmission power gain, sometimes called the transducer gain, $|S_{21}(j\omega)|^2$, satisfies

$$|S_{21}(j\omega)|^2 = 4\frac{R_1}{R_2}|H(j\omega)|^2 , \tag{9.2}$$

when it is terminated in R_1 at port-1 and in R_2 at port-2.

Darlington solved this problem in 1939, as did Cauer, and possibly others, around the same time [19, 16]. Darlington showed that a systematic procedure exists that will find a two-port that meets the requirements. Moreover it is lossless. In principle, any two-port will do as long as the transmission power gain equals $4R_1/R_2|H(j\omega)|^2$. There are practical reasons why a lossless two-port is preferred:

1. Two-ports are used mostly in communication circuits as filters. In the passband, as much of the signal power at port-1 should be transmitted to the load as possible. In the stop-band, as much of the signal power should be reflected from the two-port as possible. These considerations call for the two-port to appear as a direct connection between port-1 and port-2 over the passband, and for the input impedance to be purely imaginary in the stop-band. To meet these requirements, the two-port should be lossless.
2. Lossless two ports do not generate any resistive noise.

Unfortunately, Darlington's synthesis leads to circuits containing ideal transformers or coupled inductors with unity coefficient of coupling. These elements are impractical. However, under certain conditions, most ordinary filters can be realized as a lossless ladder terminated in resistances, without transformers. We will concentrate our study on filters of this type.

The synthesis procedure itself consists of the following key steps:

1. Obtain $|S_{21}(j\omega)|^2$ from $|H(j\omega)|^2$ and invoke the unitary property of scattering parameters of a lossless two-port to get the reflection coefficient $S_{11}(s)$ from:

$$S_{11}(s)S_{11}(-s) = 1 - |S_{21}(j\omega)|^2_{\omega^2=-s^2} . \tag{9.3}$$

2. Factor $S_{11}(s)S_{11}(-s)$. Assign all the left-half plane poles to $S_{11}(s)$. The zeros can be assigned arbitrarily and each choice leads to a different $S_{11}(s)$.
3. For each choice, there is an associated input impedance at port-1 given by:

$$Z(s) = R_1\frac{1+S_{11}(s)}{1-S_{11}(s)} . \tag{9.4}$$

4. Synthesize $Z(s)$ as the input impedance of a lossless two-port terminated in R_2 in such a way that the zeros of $S_{21}(s)$ are realized in the process.

9.2 Preliminaries

Before we present the synthesis procedure, we need a few theorems about positive real functions. The following theorem appears in Chapter 5 as Theorem 5.14.

Theorem 9.1. *Let*

$$Z(s) = \frac{m_1 + n_1}{m_2 + n_2} \tag{9.5}$$

be a positive real function, where m_1 (m_2) is the even part and n_1 (n_2) the odd part of the numerator (denominator) polynomial. Then

$$Z'(s) = \frac{m_1 + n_2}{m_2 + n_1} \tag{9.6}$$

is also a positive real function.

We recall from Sect. 4.5.1 that a polynomial $P(s) = a_n s^n + \cdots + a_1 s + a_0 = m + n$ is a Hurwitz polynomial if and only if all the zeros of $P(s)$ lie in the left-half s plane. The following theorem is well-known to students of control systems.

Theorem 9.2. *A necessary and sufficient condition that a polynomial $P(s) = m + n$ of order p is Hurwitz is that the continued fraction expansion of m/n or n/m about $s = \infty$ yields positive real coefficients and that the expansion does not terminate prematurely, i.e.*

$$\frac{m}{n} = C_1 s + \cfrac{1}{C_2 s + \cfrac{1}{C_2 s + \cfrac{1}{\ddots + \cfrac{1}{C_p s}}}} \tag{9.7}$$

with $C_1, C_2, \cdots, C_p > 0$.

In circuit theory terms, this theorem is equivalent to the following:

Theorem 9.3. *Let $P(s) = m + n$ be a Hurwitz polynomial. Then m/n or n/m is an LC impedance function.*

The proof is found in [28, 68]. From Theorem 9.1 together with this theorem, we have

Theorem 9.4. *Let*

$$Z(s) = \frac{m_1 + n_1}{m_2 + n_2} \tag{9.8}$$

be a positive real function. Then

$$Z_1(s) = \frac{m_1}{n_1}, \quad Z_2(s) = \frac{m_1}{n_2}, \quad Z_3(s) = \frac{m_2}{n_2}, \quad \text{and} \quad Z_4(s) = \frac{m_2}{n_1} \tag{9.9}$$

are all LC impedance functions.

Theorem 9.5. [1] *Let*

$$Z(s) = \frac{m_1 + n_1}{m_2 + n_2} \tag{9.10}$$

be a positive real function. Then the zeros of the ensignant $E(s^2) = m_1 m_2 - n_1 n_2$ must be such that

1. *real zeros on the σ-axis are symmetrically placed;*
2. *complex zeros have quadrantal symmetry; and*
3. *zeros on the $j\omega$-axis occur in even order.*

Proof. See Theorem 5.15.

Lastly, we need a theorem on residue conditions (Theorem 8.12) that we considered in the last chapter. It is repeated here for reference.

Theorem 9.6. *Let $z_{11}(s)$, $z_{12}(s)$ and $z_{22}(s)$ be the open-circuit impedance parameters of a lossless two-port. Then*

$$z_{11}(s) = k_{\infty(11)} s + \frac{k_{0(11)}}{s} + \sum_i \frac{2k_{i(11)} s}{s^2 + \omega_i^2}\,, \tag{9.11}$$

$$z_{12}(s) = k_{\infty(12)} s + \frac{k_{0(12)}}{s} + \sum_i \frac{2k_{i(12)} s}{s^2 + \omega_i^2}\,, \tag{9.12}$$

$$z_{22}(s) = k_{\infty(22)} s + \frac{k_{0(22)}}{s} + \sum_i \frac{2k_{i(22)} s}{s^2 + \omega_i^2}\,, \tag{9.13}$$

in which

$$k_{p(11)} k_{p(22)} - \left(k_{p(12)}\right)^2 \geq 0 \quad \textit{for each } p \tag{9.14}$$

9.3 Input impedance and two-port parameters

Assume we have completed the first three synthesis steps of Sect. 9.1. Our problem is to synthesize $Z(s)$ as the input impedance of a lossless two-port terminated in R_2. We need a few more preliminary steps.

9.3.1 Open-circuit parameters

To emphasize the role played by R_1, let

$$Z(s) = R_1 \frac{m_1 + n_1}{m_2 + n_2}\,. \tag{9.15}$$

[1] Same as Theorem 5.15, repeated here for convenience.

Expressing $Z(s)$ in terms of the open-circuit impedance parameters of the two-port and re-writing Eq. (9.15), we have

$$Z(s) = z_{11}\,\frac{1+\dfrac{z_{11}z_{22}-z_{12}^2}{R_2 z_{11}}}{1+\dfrac{z_{22}}{R_2}} = R_1\,\frac{m_1}{n_2}\,\frac{1+\dfrac{n_1}{m_1}}{1+\dfrac{m_2}{n_2}}\,. \tag{9.16}$$

We make the following identifications

$$z_{11} = R_1\,\frac{m_1}{n_2}\,, \qquad z_{22} = R_2\,\frac{m_2}{n_2}\,. \tag{9.17}$$

By Theorem 9.4, both are *LC* impedance functions, as required. We also find

$$z_{12} = \sqrt{R_1R_2}\,\frac{\sqrt{m_1m_2-n_1n_2}}{n_2}\,. \tag{9.18}$$

Alternately, the following identifications are also possible.

$$z_{11} = R_1\,\frac{n_1}{m_2}\,, \qquad z_{22} = R_2\,\frac{n_2}{m_2}\,, \tag{9.19}$$

$$z_{12} = \sqrt{R_1R_2}\,\frac{\sqrt{n_1n_2-m_1m_2}}{m_2}\,. \tag{9.20}$$

The choice depends on whether $\sqrt{\pm(m_1m_2-n_1n_2)}$ is even or odd as $z_{12}(s)$ must be odd.

9.3.2 Residue condition

The three parameters all have the same poles. Moreover, the residue condition is satisfied with an equal sign, as can be seen in the following. Suppose k_{11} is the residue of z_{11} at a pole $j\omega_0 \neq j\infty$. Then

$$k_{11} = R_1\,\frac{m_1}{n_2}\,(s-j\omega_0)\Big|_{s=j\omega_0} = R_1\,\frac{m_1}{\dfrac{dn_2}{ds}}\Bigg|_{s=j\omega_0}\,. \tag{9.21}$$

Similarly, the residue of z_{22} is

$$k_{22} = R_2\,\frac{m_2}{\dfrac{dn_2}{ds}}\Bigg|_{s=j\omega_0}\,, \tag{9.22}$$

and that of z_{12} is

$$k_{12} = \sqrt{R_1 R_2} \left. \frac{\sqrt{m_1 m_2 - n_1 n_2}}{\frac{d n_2}{ds}} \right|_{s=j\omega_0} . \tag{9.23}$$

Making substitutions, we find

$$k_{11} k_{22} - k_{12}^2 = 0 . \tag{9.24}$$

If the parameters have a pole at ∞, then they can be expressed as

$$z_{11}(s) = k_{\infty(11)} s + \cdots \quad z_{12}(s) = k_{\infty(12)} s + \cdots \quad z_{22}(s) = k_{\infty(22)} s + \cdots . \tag{9.25}$$

The coefficients (they are not residues, strictly speaking) are found from

$$k_{\infty(11)} = R_1 \left. \frac{m_1}{n_2} \frac{1}{s} \right|_{s=\infty} , \tag{9.26}$$

$$k_{\infty(12)} = \sqrt{R_1 R_2} \left. \frac{\sqrt{m_1 m_2 - n_1 n_2}}{n_2} \frac{1}{s} \right|_{s=\infty} , \tag{9.27}$$

$$k_{\infty(22)} = R_2 \left. \frac{m_2}{n_2} \frac{1}{s} \right|_{s=\infty} . \tag{9.28}$$

The residue condition becomes

$$k_{\infty(11)} k_{\infty(22)} - k_{\infty(12)}^2 = R_1 R_2 \left. \frac{n_1}{n_2} \frac{1}{s^2} \right|_{s=\infty} = 0 , \tag{9.29}$$

because $\deg(n_1) \leq \deg(n_2)$. Similar consideration applies to the case in which $z_{11}(s) = R_1 n_1 / m_2$.

We conclude that the residue condition holds with an equal sign. This implies that if we have somehow realized two of the three parameters, the third one is automatically realized.

9.3.3 Auxiliary polynomial

We should remind ourselves that z_{12} is supposed to be a rational function of s. It is possible that $m_1 m_2 - n_1 n_2$ is not a perfect square. In that case, we multiply the numerator and denominator of $Z(s)$ by a polynomial $m_0 + n_0$ to get

$$Z(s) = R_1 \frac{m_1 + n_1}{m_2 + n_2} \times \frac{m_0 + n_0}{m_0 + n_0} = R_1 \frac{\hat{m}_1 + \hat{n}_1}{\hat{m}_2 + \hat{n}_2} . \tag{9.30}$$

The even polynomial $\hat{E}(s^2) = \hat{m}_1 \hat{m}_2 - \hat{n}_1 \hat{n}_2$ is

$$\hat{E}(s^2) = (m_1 m_2 - n_1 n_2)(m_0^2 - n_0^2) = E(s^2)(m_0^2 - n_0^2)\,. \tag{9.31}$$

By Theorem 9.5, real roots of $E(s^2)$ of odd order come from a factor such as $(s_k - s)(s_k + s)$. For each such factor, we include $(s_k + s)$ in the polynomial $m_0 + n_0$. A complex root $re^{j\theta}$ of odd order comes from a factor $(r^2 + 2rs\cos\theta + s^2)(r^2 - 2rs\cos\theta + s^2)$. For each such factor, we include $(r^2 + 2rs\cos\theta + s^2)$ in $m_0 + n_0$. Roots on the $j\omega$-axis occur in even order and need not be included in $m_0 + n_0$. It is thus always possible to find a polynomial $m_0 + n_0$ such that $\hat{E}(s^2)$ is a perfect square. We call $m_0 + n_0$ an *auxiliary polynomial.*

9.3.4 Short-circuit parameters

In a dual manner, we can also relate the input admittance function $Y(s) = 1/Z(s)$ to the short-circuit parameters, as follows.

$$Y(s) = y_{11}\,\frac{1 + \dfrac{y_{11}y_{22} - y_{12}^2}{G_2 y_{11}}}{1 + \dfrac{y_{22}}{G_2}} = G_1\,\frac{m_2}{n_1}\,\frac{1 + \dfrac{n_2}{m_2}}{1 + \dfrac{m_1}{n_1}}\,. \tag{9.32}$$

We make the following identifications

$$\begin{aligned} y_{11} &= G_1\,\frac{m_2}{n_1}\,, & y_{22} &= G_2\,\frac{m_1}{n_1}\,,\\ y_{12} &= \sqrt{G_1 G_2}\,\frac{\sqrt{m_1 m_2 - n_1 n_2}}{n_1}\,. \end{aligned} \tag{9.33}$$

Alternately,

$$\begin{aligned} y_{11} &= G_1\,\frac{n_2}{m_1}\,, & y_{22} &= G_2\,\frac{n_1}{m_1}\,,\\ y_{12} &= \sqrt{G_1 G_2}\,\frac{\sqrt{n_1 n_2 - m_1 m_2}}{m_1}\,, \end{aligned} \tag{9.34}$$

where $G_1 = 1/R_1$ and $G_2 = 1/R_2$.

9.3.5 Scattering parameters

It is instructive to express the transmission function $S_{21}(s)$ in terms of the $m's$ and $n's$. With reference to Fig. 9.1, solving for the transfer function in terms of the open-circuit parameters, we get

$$\frac{V_2}{E} = \frac{R_2 z_{12}}{R_1(z_{22}+R_2)+z_{11}z_{22}-z_{12}^2+z_{11}R_2}. \tag{9.35}$$

After substitutions, we find the transmission function given by

$$S_{21}(s) = 2\,\frac{\sqrt{m_1m_2-n_1n_2}}{m_1+m_2+n_1+n_2}. \tag{9.36}$$

Thus the transmission zeros are the same as the zeros of z_{12} and they can be rendered to have an even order, using an auxiliary common factor m_0+n_0 if necessary. The transmission power gain is

$$\begin{aligned}
|S_{21}(j\omega)|^2 &= 4\left.\frac{m_1m_2-n_1n_2}{(m_1+m_2)^2-(n_1+n_2)^2}\right|_{s=j\omega} \\
&= \left|1-\frac{(m_1-m_2)^2-(n_1-n_2)^2}{(m_1+m_2)^2-(n_1+n_2)^2}\right|_{s=j\omega} \\
&= 1-\left|\frac{R_1\dfrac{m_1+n_1}{m_2+n_2}-R_1}{R_1\dfrac{m_1+n_1}{m_2+n_2}+R_1}\right|^2_{s=j\omega} = 1-|S_{11}(j\omega)|^2,
\end{aligned} \tag{9.37}$$

as expected. So everything is consistent.

9.3.6 Transmission zeros

Assume we have an impedance function $Z(s)$ such that the zeros of its ensignant $E(s^2)=m_1m_2-n_1n_2$ are all of even order. We want to synthesize $Z(s)$ as the input impedance of a lossless two-port terminated in R_2. We look for a circuit consisting of cascade of two-port sections such that each section will produce a zero of $z_{12}(s)$ or $S_{21}(s)$ as we synthesize $Z(s)$. It is remarkable that such a circuit exists.

Let us call the zeros of $S_{21}(s)$ the *transmission zeros*. They can be classified into three groups.

1. Transmission zeros on the $j\omega$-axis, including the points 0 and $j\infty$;
2. Transmission zeros on the real axis; and
3. Complex transmission zeros.

We will concentrate mainly on zeros of group (1) because of their practical importance.

9.4 Synthesis of two-port with imaginary transmission zeros

This class includes all practical filter circuits: low-pass, high-pass, bandpass, and band-elimination. Since by frequency transformation, the latter three can all be derived from the first, we will further concentrate on synthesis of low-pass filters.

In a low-pass filter, all the transmission zeros are at infinity or at finite frequencies. A transmission zero at infinity can be realized by a shunt capacitance or series inductor section as shown in Fig. 9.2**a** and **b**. A transmission zero at finite frequency can be realized by a section of series *LC* in shunt or a section of parallel *LC* as shown in Fig. 9.2**c** and **d**. We will illustrate the synthesis procedures by examples.

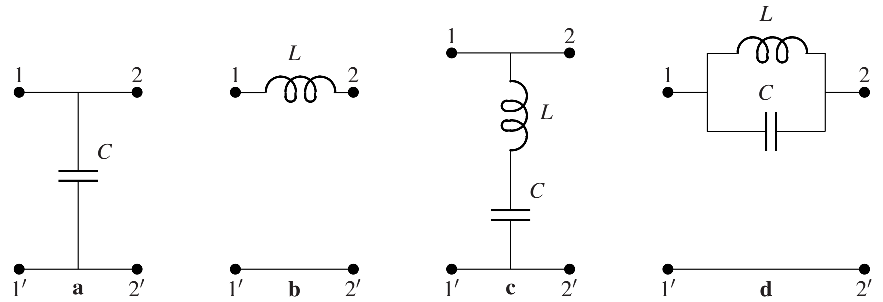

Fig. 9.2 A transmission zero at infinity is realized in section **a** or **b**. A transmission zero at finite frequency is realized in section **c** or **d**

9.4.1 Transmission zeros all at infinity

Example 9.1. Find a two-port terminated in $R_1 = 1\ \Omega$ and $R_2 = 4\ \Omega$ that realizes the following transfer function in magnitude squared:

$$|H(j\omega)|^2 = \frac{A^2}{64\omega^{10} - 160\omega^8 + 140\omega^6 - 50\omega^4 + 6.25\omega^2 + 1}, \tag{9.38}$$

which is plotted in Fig. 9.3. It is seen the frequency response is that of a low-pass filter, in fact a Chebyshev filter with equi-ripple characteristics in the passband. The constant is best determined in this instance at DC. We have

$$|H(0)|^2 = A^2 = \left|\frac{V}{E}\right|^2_{\omega=0} = \frac{R_2^2}{(R_1 + R_2)^2} = \frac{16}{25}. \tag{9.39}$$

The next to the last expression follows from the fact that there is a direct connection between the source and the load at DC in a low-pass filter.

We begin with the transmission power gain obtained from

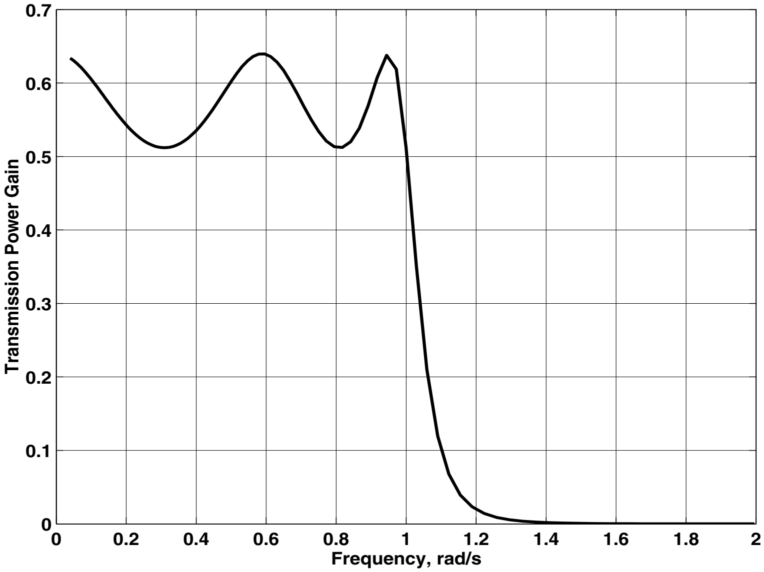

Fig. 9.3 Frequency response of a Chebyshev low-pass filter with $R_1 = 1\,\Omega$ and $R_2 = 4\,\Omega$

$$
\begin{aligned}
|S_{21}(j\omega)|^2 &= 4\frac{R_1}{R_2}\,|H(j\omega)|^2 \\
&= \frac{16}{25}\,\frac{1}{64\omega^{10} - 160\omega^8 + 140\omega^6 - 50\omega^4 + 6.25\omega^2 + 1}\,.
\end{aligned}
\tag{9.40}
$$

We note that $|S_{21}(j\omega)|^2 \leq 1$ for all ω, as required. The polynomial $E(s^2) = m_1 m_2 - n_1 n_2$ is a constant and there are five transmission zeros, all at infinity.

Step 1. Find $S_{11}(s)S_{11}(-s)$.

$$
\begin{aligned}
&S_{11}(j\omega)S_{11}(-j\omega) = 1 - |S_{21}(j\omega)|^2 \\
&= \frac{1600\omega^{10} - 4000\omega^8 + 3500\omega^6 - 1250\omega^4 + 156.25\omega^2 + 9}{1600\omega^{10} - 4000\omega^8 + 3500\omega^6 - 1250\omega^4 + 156.25\omega^2 + 25}\,,
\end{aligned}
\tag{9.41}
$$

from which we obtain $S_{11}(s)S_{11}(-s)$ by replacing ω^2 by $-s^2$. In factored form, we have[2]

$$
\begin{aligned}
S_{11}(s)S_{11}(-s) = &\frac{(s+z_1)(s^2+z_2 s+z_3)(s^2+z_4 s+z_5)}{(s+p_1)(s^2+p_2 s+p_3)(s^2+p_4 s+p_5)} \\
&\times \frac{(-s+z_1)(s^2-z_2 s+z_3)(s^2-z_4 s+z_5)}{(-s+p_1)(s^2-p_2 s+p_3)(s^2-p_4 s+p_5)}\,.
\end{aligned}
\tag{9.42}
$$

[2] The MATLAB®function *r=roots(p)* obtains a column vector of roots of polynomial p expressed as a row vector of coefficients in descending order.

$$z_1 = 0.2046, \quad z_2 = 0.1264, \quad z_3 = 0.9464, \quad z_4 = 0.3310, \quad z_5 = 0.3874.$$
$$p_1 = 0.2928, \quad p_2 = 0.1809, \quad p_3 = 0.9902, \quad p_4 = 0.4737, \quad p_5 = 0.4312.$$

Step 2. Assign all the left-half plane poles to $S_{11}(s)$. The zeros can be assigned in many different ways. Let us do two cases. Case (a): Assign all the left-half plane zeros to $S_{11(a)}(s)$. Case (b): Assign all the right-half plane zeros to $S_{11(b)}(s)$. [3]

$$S_{11(a)}(s) = \pm \frac{s^5 + 0.6621s^4 + 1.4692s^3 + 0.6437s^2 + 0.4407s + 0.0750}{s^5 + 0.9474s^4 + 1.6988s^3 + 0.9883s^2 + 0.5871s + 0.1250}. \quad (9.43)$$

$$S_{11(b)}(s) = \pm \frac{-s^5 + 0.6621s^4 - 1.4692s^3 + 0.6437s^2 - 0.4407s + 0.0750}{s^5 + 0.9474s^4 + 1.6988s^3 + 0.9883s^2 + 0.5871s + 0.1250}. \quad (9.44)$$

To determine the sign, we note that at DC the input impedance $Z_{in(1)}(0) = R_2$ and

$$S_{11}(0) = \frac{R_2 - R_1}{R_2 + R_1} = 0.6 > 0 \quad (9.45)$$

So we must use the plus sign.

Step 3. Each choice of $S_{11}(s)$ leads to a different impedance function:

$$Z_a(s) = R_1 \frac{1 + S_{11(a)}(s)}{1 - S_{11(a)}(s)} = \frac{2s^5 + 1.6095s^4 + 3.1679s^3 + 1.6320s^2 + 1.0278s + 0.2}{0.2853s^4 + 0.2296s^3 + 0.3446s^2 + 0.1464s + 0.05}. \quad (9.46)$$

$$Z_b(s) = R_1 \frac{1 + S_{11(b)}(s)}{1 - S_{11(b)}(s)} = \frac{1.6095s^4 + 0.2296s^3 + 1.6320s^2 + 0.1464s + 0.2}{2s^5 + 0.2853s^4 + 3.1680s^3 + 0.3446s^2 + 1.0278s + 0.05}. \quad (9.47)$$

Step 4. Take $Z_a(s)$ first. Since the numerator of $|S_{12}(j\omega)|^2$ is a constant, by Eq. (9.36), $\sqrt{\pm(m_1 m_2 - n_1 n_2)}$ is even, and we make the following identifications:

$$z_{11}(s) = R_1 \frac{m_1}{n_2} = \frac{1.6095s^4 + 1.6320s^2 + 0.2}{0.2296s^3 + 0.1464s}, \quad (9.48)$$

$$z_{22}(s) = R_2 \frac{m_2}{n_2} = 4\,\frac{0.2853s^4 + 0.3446s^2 + 0.05}{0.2296s^3 + 0.1464s}, \quad (9.49)$$

$$z_{12}(s) = \sqrt{R_1 R_2} \frac{\sqrt{m_1 m_2 - n_1 n_2}}{n_2} = \frac{0.2}{0.2296s^3 + 0.1464s}. \quad (9.50)$$

Step 5. We realize $z_{11}(s)$ by continued fraction expansion about $s = \infty$ to produce, in this case, four transmission zeros of $S_{21}(s)$. The fifth one is produced by $z_{22}(s)$ as its private pole at $s = \infty$. In order not to miss any private poles, it is suggested [68] that we realize $z_{11}(s)$ from port-1 and $z_{22}(s)$ from port-2 simultaneously, meeting half-way, where we can check our answers. In continued fraction expan-

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[3] The MATLAB®function *p=poly(r)* obtain the coefficients of polynomial p in descending order in a row vector whose roots are placed in a column vector r.

sion, we make subtractions at each stage and we lose significant digits quickly as we progress. By synthesizing the 2-port from both ends, we reduce the number of stages, thereby avoiding round-off errors. The final circuit is shown in Fig. 9.4.

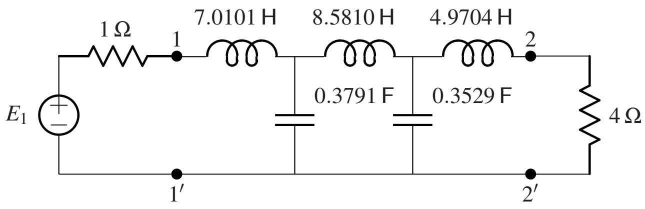


Fig. 9.4 A fifth order Chebyshev low-pass filter

In a similar manner, from $Z_b(s)$ we make the identifications:

$$z_{11}(s) = \frac{1.6095s^4 + 1.6320s^2 + 0.2}{2s^5 + 3.1680s^3 + 1.0278s}, \tag{9.51}$$

$$z_{22}(s) = 4\,\frac{0.2853s^4 + 0.3446s^2 + 0.05}{2s^5 + 3.1680s^3 + 1.0278s}, \tag{9.52}$$

$$z_{12}(s) = \frac{0.2}{2s^5 + 3.1680s^3 + 1.0278s}. \tag{9.53}$$

Expand $1/z_{11}(s)$ and $1/z_{22}(s)$ about $s = \infty$ and we get another circuit, shown in Fig. 9.5. The two circuits have the same transmission power gain. The second is preferred because it uses fewer inductors.

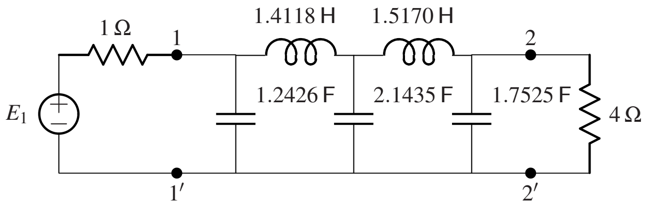


Fig. 9.5 Another fifth order Chebyshev low-pass filter with the same transmission power gain as that of the filter of Fig. 9.4

9.4.2 Transmission zeros at finite frequencies

Example 9.2. Find a two-port terminated in a one-ohm resistor at both ends that has the following transmission power gain:

$$|S_{21}(j\omega)|^2 = G(\omega^2) = \frac{(1-\omega^2)^2}{225\omega^6 - 35\omega^4 + 0.25\omega^2 + 1}. \tag{9.54}$$

It has a pair of transmission zeros at $s = \pm j\omega_1 = \pm j$ and at infinity. The even polynomial $E(s^2) = m_1m_2 - n_1n_2 = (s^2+1)^2$.

Following the same procedure as in the last example, we obtain

$$|S_{11}(j\omega)|^2 = \left.\frac{-s^2(s^2+0.2s+0.1)(s^2-0.2s+0.1)}{(s^3+\frac{2}{3}s^2+0.3s+\frac{1}{15})(-s^3+\frac{2}{3}s^2-0.3s+\frac{1}{15})}\right|_{s^2=-\omega^2}. \tag{9.55}$$

We form an $S_{11}(s)$ such that the input impedance function $Z(s)$ will have a zero at infinity in order to minimize the number of inductors in the realization. Accordingly, we choose

$$S_{11}(s) = \frac{-s(s^2+0.2s+0.1)}{s^3+0.6667s^2+0.3s+0.0667}, \quad \text{and} \tag{9.56}$$

$$Z(s) = \frac{7s^2+3s+1}{30s^3+13s^2+6s+1}. \tag{9.57}$$

Noting that $\sqrt{m_1m_2 - n_1n_2}$ is even, we make the following identifications:

$$z_{11}(s) = \frac{7s^2+1}{30s^3+6s}, \tag{9.58}$$

$$z_{12}(s) = \frac{s^2+1}{30s^3+6s}, \tag{9.59}$$

$$z_{22}(s) = \frac{13s^2+1}{30s^3+6s}. \tag{9.60}$$

Let $y = 1/z_{11}$. It has a pole at infinity. Remove a capacitor C_1 such that the remainder admittance y_1 has zeros at $s = \pm j$, namely

$$y_1(j) = y(j) - jC_1 = 0, \tag{9.61}$$

and we find $C_1 = 4$ F. The remainder is

$$y_1(s) = y(s) - C_1 s = \frac{(s^2+1)2s}{7s^2+1}. \tag{9.62}$$

Removing the poles of $z_1(s) = 1/y_1(s)$ at $s = \pm j$ to realize the transmission zeros, we get

$$z_1(s) = \frac{1}{y_1(s)} = \frac{7s^2+1}{(s^2+1)2s} = \frac{3s}{s^2+1} + \frac{1}{2s}. \tag{9.63}$$

The remainder is a capacitor of 2 F. Terminate port-2 with a one-ohm resistor and we obtain a realization shown in Fig. 9.6.

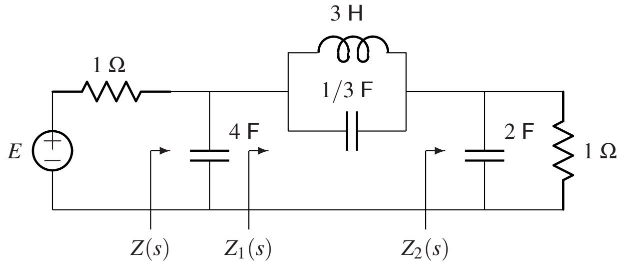


Fig. 9.6 A low pass filter with a transmission zero at a finite frequency

As we synthesize $z_{11}(s)$, we realize the transmission zeros at $s = \pm j\omega_1$ by removing a shunt capacitor and a parallel LC in the series arm, leaving a remainder $z_2(s)$ which is, by the removal theorem 5.19, a positive real function and two degrees lower in order. The same procedure is then applied to $z_2(s)$ until there is no remainder. We shall call the shunt-capacitor-parallel-LC combination a *$j\omega$-zero-section* depicted in Fig. 9.7, to signify it produces a pair of transmission zeros on the imaginary axis.

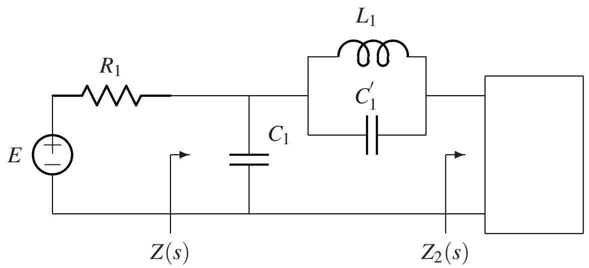


Fig. 9.7 The combination $C_1 - L_1 - C_1^{'}$ constitutes a $j\omega_1$-zero-section which produces a pair of transmission zeros at $s = \pm j\omega_1$

The procedure can be summarized in an expression for $z_{11}(s)$:

$$z_{11}(s) = \cfrac{1}{C_1 s + \cfrac{1}{\cfrac{2k_1 s}{s^2 + \omega_1^2} + Z_2(s)}} \tag{9.64}$$

where, with reference to Fig. 9.7

$$C_1 = \frac{1}{j\omega_1 z_{11}(j\omega_1)}, \qquad C_1^{'} = \frac{1}{2k_1}, \qquad L_1 = \frac{2k_1}{\omega_1^2}, \tag{9.65}$$

where k_1 is the residue of $z_1(s) = 1/(1/z_{11} - C_1 s)$ at $s = j\omega_1$.

Example 9.3. Find a lossless ladder terminated in a one-ohm resistor at both ends whose transmission power gain is:

$$|S_{21}(j\omega)|^2 = \frac{4(1-\omega^2)^2(4-\omega^2)^2}{a_{12}\omega^{12} + a_{10}\omega^{10} + a_8\omega^8 + a_6\omega^6 + a_4\omega^4 + a_2\omega^2 + a_0}\,. \tag{9.66}$$

$$\begin{array}{llll} a_{12} = 26569\,, & a_{10} = -81523\,, & a_8 = 76618\,, & a_6 = -17671\,, \\ a_4 = 1700\,, & a_2 = -96\,, & a_0 = 64\,. & \end{array}$$

There are three transmission zeroes: $s = j$, $s = j2$ and $s = \infty$. Following the same procedure as before, we find the impedance function at port-1 to be:

$$Z(s) = \frac{38s^5 + 38s^4 + 76s^3 + 45s^2 + 20s + 4}{163s^6 + 163s^5 + 336s^4 + 203s^3 + 105s^2 + 28s + 4}\,. \tag{9.67}$$

It is chosen so as to minimize the number of inductors in the realization. Let us carry out the synthesis on an admittance basis. Let

$$y_{11}(s) = \frac{m_2}{n_1} = \frac{163s^6 + 336s^4 + 105s^2 + 4}{38s^5 + 76s^3 + 20s}\,. \tag{9.68}$$

It has a pole at $s = \infty$. Removing a capacitor C_1 to leave a remainder $y_1(s)$ such that $y_1(j) = 0$, we get

$$C_1 = 4\,. \tag{9.69}$$

The reciprocal of the remainder, $z_1(s)$, has poles at $s = \pm j$:

$$z_1(s) = \frac{3s}{s^2+1} + \frac{5s^3 + 8s}{11s^4 + 21s^2 + 4} \tag{9.70}$$

After removing a parallel LC, we get a remainder $z_2(s)$ whose reciprocal $y_2(s)$ is

$$y_2(s) = \frac{11s^4 + 21s^2 + 4}{5s^3 + 8s}\,. \tag{9.71}$$

Remove a capacitor C_2 to create a zero-section at $s = j2$:

$$C_2 = \frac{y_2(j2)}{j2} = 2\,. \tag{9.72}$$

The reciprocal of the remainder, $z_3(s)$, is

$$z_3(s) = \frac{4s}{s^2+4} + \frac{s}{s^2+1}\,. \tag{9.73}$$

Removal of an LC parallel circuit leaves a remainder

$$z_4(s) = \frac{s}{s^2+1}\,. \tag{9.74}$$

Its reciprocal $y_4(s)$ has a pole at ∞. Removing it and we have a remainder which is an inductor, placed in the series arm to realize the transmission zero at ∞. The final realization is shown in Fig. 9.8.

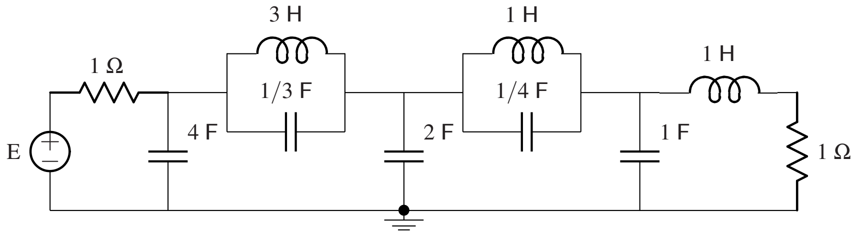


Fig. 9.8 A lossless ladder realization of a transfer function with transmission zeros at $\omega = 1, 2, \infty$

9.4.3 Order of removal of transmission zeros

In the last example, we removed a zero-section to realize the transmission zero at $s = \pm j$ first and then at $s = \pm j2$ next. Does it matter in which order the zero-sections are removed? That it does is illustrated in Fig. 9.9 in which the zero-section at $s = \pm j2$ is removed first, followed by that at $s = \pm j$. The circuit is not realizable as it has a negative capacitance at the end.

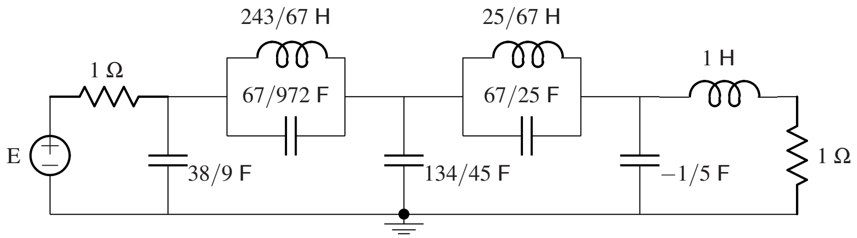


Fig. 9.9 Realization of transmission zero at $\omega = 2$ before the zero at $\omega = 1$ results in an non-physical circuit

In 1955, Fujisawa studied this problem [27] and showed that if we are to avoid the use of ideal transformers in a lossless ladder to realize a low-pass transfer function with transmission zeros $\{\omega_1, \omega_2, \ldots, \omega_n, \infty\}$, then the *next* capacitor to remove must be such that

$$C = \min_{\omega_k} \left[\frac{y_{11}(j\omega_1)}{j\omega_1}, \frac{y_{11}(j\omega_2)}{j\omega_2}, \cdots, \frac{y_{11}(j\omega_n)}{j\omega_n}, \lim_{s\to\infty} \frac{y_{11}(s)}{s} \right], \tag{9.75}$$

and this rule is applied recursively. In our example,

$$C = \min \left[\frac{y_{11}(j)}{j}, \frac{y_{11}(j2)}{j2}, \frac{163}{38} \right] = \min\,[4.0, 4.222, 4.289]\,. \tag{9.76}$$

So we must realize the transmission zeros at $\omega = 1$ first.

9.5 Brune section

If coupled inductors with unity coupling coefficient are allowed, then transfer functions with transmission zeros on the $j\omega$-axis can be realized as cascade of Brune sections. This is not surprising since the real part of the input impedance is zero at the transmission zeros on the $j\omega$-axis, and we have shown that after removal of a zero-section, the remaining transmission zeros are intact.

We recall that each Brune section (Sect. 7.1) starts with an impedance function which is minimum-reactance and minimum susceptance and its real part has a double zero at, say, ω_0, on the $j\omega$-axis. We first remove an inductance to leave a remainder with a zero at ω_0. After removal of a shunt admittance of LC in series to realize the zero, we remove an inductance from the remainder impedance in order to cancel the pole at ∞ we artificially introduced at the beginning. This device calls for a negative inductor which is later absorbed in a system of coupled inductors with unity coupling coefficient.

If it happens that there is an inductance outside a Brune section which is in series with the negative inductance, the two can be combined and a perfect coupled inductor is not required.

Example 9.4. Let us synthesize the transmission function of Eq. (9.66) whose associated impedance function is given in Eq. (9.67), repeated below.

$$Z(s) = \frac{38s^5 + 38s^4 + 76s^3 + 45s^2 + 20s + 4}{163s^6 + 163s^5 + 336s^4 + 203s^3 + 105s^2 + 28s + 4}\,. \tag{9.77}$$

As it stands, it is not minimum-susceptance or minimum-reactance. First we remove a pole of its reciprocal $Y(s) = 1/Z(s)$ at ∞ to leave a remainder $Y_1(s)$:

$$Y_1(s) = Y(s) - \frac{163}{38}s = \frac{10s^4 + \frac{379}{38}s^3 + \frac{730}{38}s^2 + \frac{412}{38}s + 4}{38s^5 + 38s^4 + 76s^3 + 45s^2 + 20s + 4}\,. \tag{9.78}$$

Its reciprocal $Z_1(s) = 1/Y_1(s)$ has a pole at ∞. Remove it and the remainder is:

$$Z_2(s) = Z_1(s) - \frac{19}{5}s = \frac{\frac{1}{10}s^4 + 3s^3 + \frac{19}{5}s^2 + \frac{24}{5}s + 4}{10s^4 + \frac{379}{38}s^3 + \frac{730}{38}s^2 + \frac{412}{38}s + 4}. \tag{9.79}$$

It is minimum-susceptance and minimum-reactance. Moreover, its real part has zeros at $\omega_1 = \pm 1$ and $\omega_2 = \pm 2$, as we expect. We could proceed as we did before by following Brune's procedure to synthesize $Z_2(s)$. Instead, let us do so using two-port impedance parameters. Let

$$Z_2(s) = \frac{m_1 + n_1}{m_2 + n_2} \tag{9.80}$$

and we make the following identifications:

$$z_{11}(s) = \frac{m_1}{n_2} = \frac{\frac{1}{10}s^4 + \frac{19}{5}s^2 + 4}{\frac{379}{38}s^3 + \frac{412}{38}s}, \tag{9.81}$$

$$z_{12}(s) = \frac{\sqrt{m_1 m_2 - n_1 n_2}}{n_2} = \frac{(s^2+1)(s^2+4)}{\frac{379}{38}s^3 + \frac{412}{38}s}, \tag{9.82}$$

$$z_{22}(s) = \frac{m_2}{n_2} = \frac{10s^4 + \frac{730}{38}s^2 + 4}{\frac{379}{38}s^3 + \frac{412}{38}s}, \tag{9.83}$$

keeping in mind that any impedance in series with port-2 will only appear in z_{22} and not in z_{11}, and vice versa.

We now proceed to synthesize z_{11} in such a way that z_{12} and z_{22} are realized at the same time.

We first remove an inductance L_a from z_{11} so that the remainder impedance $z_3(s)$ will have zeros at $s = \pm j$. L_a is determined from:

$$z_3(j) = z_{11}(j) - jL_a = 0 \tag{9.84}$$

and we find

$$L_a = -\frac{19}{55}, \tag{9.85}$$

and the remainder is

$$z_3(s) = z_{11}(s) - L_a s = \frac{(s^2+1)(\frac{39}{11}s^2 + 4)}{\frac{379}{38}s^3 + \frac{412}{38}s}. \tag{9.86}$$

Expand the reciprocal:

$$y_3(s) = \frac{1}{z_3(s)} = \frac{\frac{363}{190}s}{s^2+1} + \frac{176s}{5(39s^2+44)}\,. \tag{9.87}$$

The first term is a series LC admittance in shunt, with

$$M_1 = \frac{190}{363}, \qquad C_1 = \frac{363}{190}\,. \tag{9.88}$$

From the reciprocal of the remainder admittance we must remove an inductance L_b of such value as to complete the Brune section, i.e., it must have a value to cancel the pole at ∞ that was introduced when we took out L_a. This condition is expressed as:

$$(L_a + M_1)(L_b + M) = M_2^2 \tag{9.89}$$

and we find

$$L_b = \frac{190}{187}\,. \tag{9.90}$$

Remove it and we get a remainder impedance:

$$z_5(s) = z_4(s) - L_b s = \frac{5(39s^2+44)}{176s} - \frac{190}{187}s = \frac{25s^2+340}{272s}\,. \tag{9.91}$$

The second Brune section begins here. Remove an inductance L_c from $z_5(s)$ to create a zero section at $s = j2$:

$$L_c = \frac{z_5(j2)}{j2} = -\frac{15}{68}\,. \tag{9.92}$$

The remainder impedance is:

$$z_6(s) = z_5(s) - L_c s = \frac{5(s^2+4)}{16s}\,. \tag{9.93}$$

Its reciprocal is a series LC admittance which realizes the transmission zeros at $s = \pm j2$. The element values are:

$$M_2 = \frac{16}{5}, \qquad C_2 = \frac{4}{5}\,. \tag{9.94}$$

After the removal of this zero-section, we are left with no remainder (open circuit). We must add an inductance L_d to complete the second Brune section. L_d will be in series with port-2 and does not appear in z_{11} but it is part of z_{22}. It is found from

$$(L_c + M_2)(L_d + M_2) = M_2^2\,, \tag{9.95}$$

and it is

$$L_d = \frac{3}{4}\,. \tag{9.96}$$

As a check, we note that

$$\lim_{s\to\infty}\frac{z_{22}(s)}{s} = L_d + \cfrac{1}{\cfrac{16}{5} + \cfrac{1}{-\cfrac{15}{68} + \cfrac{190}{187} + \cfrac{190}{363}}} = \frac{380}{379} \tag{9.97}$$

which is the same as that computed from Eq. (9.83). The complete circuit is shown in Fig. 9.10. The two negative inductances can be combined with the inductances to

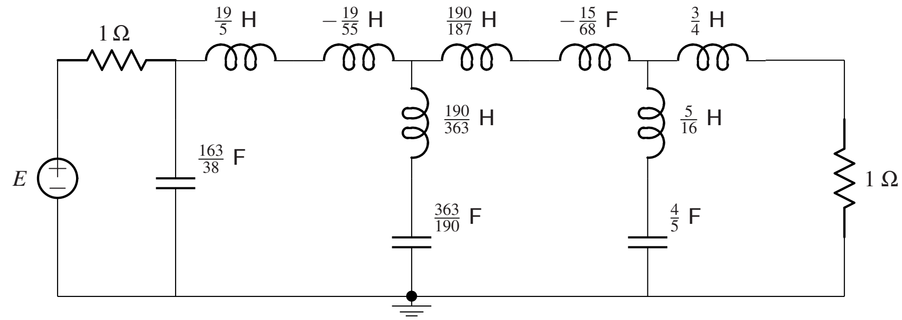


Fig. 9.10 Brune realization of a transfer function with two transmission zeros on the $j\omega$-axis

their left, to result in a circuit without coupled inductors, or with coupled inductors whose coupling coefficients are both less than one, as shown in Fig. 9.11.

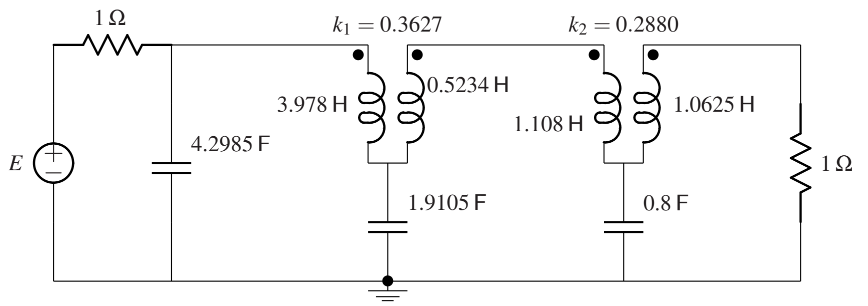


Fig. 9.11 Brune realization in which negative inductances are absorbed in inductances outside the sections

9.6 Darlington C-section

We now take up the case of synthesis of a transfer function with transmission zeros on the real axis. We recall in the Brune section, the transmission zeros are realized in a series LC admittance connected in shunt. If the inductance L in this admittance is negative, the admittance $Y(s)$ produces two zeros on the real axis:

$$Y(s) = \frac{1}{-|L|s + \dfrac{1}{Cs}} = \frac{-\dfrac{1}{|L|}s}{s^2 - \dfrac{1}{|LC|}}, \tag{9.98}$$

at

$$s = \pm\frac{1}{|LC|}. \tag{9.99}$$

The negative inductance is realized as the mutual inductance of two coupled inductors with unity coupling coefficient.

Example 9.5. Let us find a circuit which realizes the following voltage transfer function of a two-port terminated in $R_1 = 1\,\Omega$ at port-1 and in $R_2 = 1/4\,\Omega$ at port-2.

$$\left|\frac{V_2(j\omega)}{E}\right|^2 = \frac{\omega^4 + 4\omega^2 + 4}{16\omega^4 + 89\omega^2 + 100} \tag{9.100}$$

The transmission function associated with this transfer function is:

$$\begin{aligned}|S_{21}(j\omega)|^2 = 4\frac{R_1}{R_2}\left|\frac{V_2(j\omega)}{E}\right|^2 &= \frac{16\omega^4 + 64\omega^2 + 64}{16\omega^4 + 89\omega^2 + 100} \\ &= \left.\frac{(s^2-2)^2}{(s+2)(s+5/4)(-s+2)(-s+5/4)}\right|_{s=j\omega}.\end{aligned} \tag{9.101}$$

So the transmission zeros are on the real axis at $s = \pm\sqrt{2}$. Following the usual procedure, we find one of the two possible input impedance functions at port-1 to be:

$$Z(s) = \frac{s^2 + \dfrac{9}{2}s + 1}{s^2 + 2s + 4} = R_1\,\frac{m_1 + n_1}{m_2 + n_2}, \tag{9.102}$$

which is minimum-reactance and minimum-susceptance. We make the following identifications:

$$z_{11}(s) = R_1\,\frac{m_1}{n_2} = \frac{s^2+1}{2s}, \tag{9.103}$$

$$z_{12}(s) = \sqrt{R_1R_2}\,\frac{\sqrt{m_1m_2 - n_1n_2}}{n_2} = \frac{s^2-2}{4s}, \tag{9.104}$$

$$z_{22}(s) = R_2\,\frac{m_2}{n_2} = \frac{s^2+4}{8s}. \tag{9.105}$$

From z_{11} we remove an inductor L_a and leave a remainder $z_1(s)$ with zeros at $s = \pm\sqrt{2}$, i.e., we find L_a from:

$$z_1(\sqrt{2}) = z_{11}(\sqrt{2}) - \sqrt{2}L_a = 0 \tag{9.106}$$

and get

$$L_a = \frac{3}{4}, \tag{9.107}$$

and

$$z_1(s) = \frac{-s^2+2}{4s}. \tag{9.108}$$

Its reciprocal is an admittance of a negative inductance M in series with a capacitance C:

$$M = -\frac{1}{4}\,\mathsf{H}, \qquad C = 2\,\mathsf{F}. \tag{9.109}$$

The remainder is an open-circuit. We now must add an inductance L_c in series with port-2 to cancel the pole at ∞ we introduced when we removed L_a from z_{11}. It is found from:

$$(L_a + M)(L_c + M) = M^2 \tag{9.110}$$

Its value is:

$$L_c = \frac{3}{8}\,\mathsf{H} \tag{9.111}$$

As a check,

$$\lim_{s\to\infty} \frac{z_{22}}{s} = L_c + M = \frac{1}{8}, \tag{9.112}$$

which is the same as that computed from Eq. (9.105). The three inductors are realized as two coupled inductors with unity coupling coefficient and opposite directions of winding of the coils, signified by the two antisymmetric dots in Fig. 9.12.

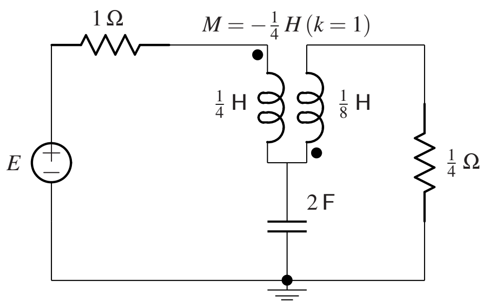


Fig. 9.12 Darlington C-section to realize a transfer function with transmission zeros on the real axis

9.7 Darlington D-section

The last case of synthesis of transfer functions is one where the transmission zeros are complex and occur in quadrantal symmetry at $s_{1,2} = \sigma_0 \pm j\omega_0$ and $s_{3,4} = -\sigma_0 \pm j\omega_0$. Following the same approach as the previous two cases, we take out an impedance from the input impedance associated with the given transmission function to create a remainder with zeros at frequencies (complex) of the transmission zeros. The resultant circuit is somewhat more complicated as it requires two sets of perfect coupled inductors. Before we present the synthesis procedure, we need to review a few properties about two-ports.

9.7.1 Two-ports and ideal transformers

1. Let $[z]$ be the open-circuit impedance matrix of a 2-port. If it can be decomposed into a sum of two realizable impedance matrices $[z^{(a)}]$ and $[z^{(b)}]$, then the 2-port can be realized as a series connection of two 2-ports, each being characterized by their respective impedance matrices, as shown in Fig. 9.13. The 1 : 1 ideal transformers are necessary to prevent one from possibly shorting the other.

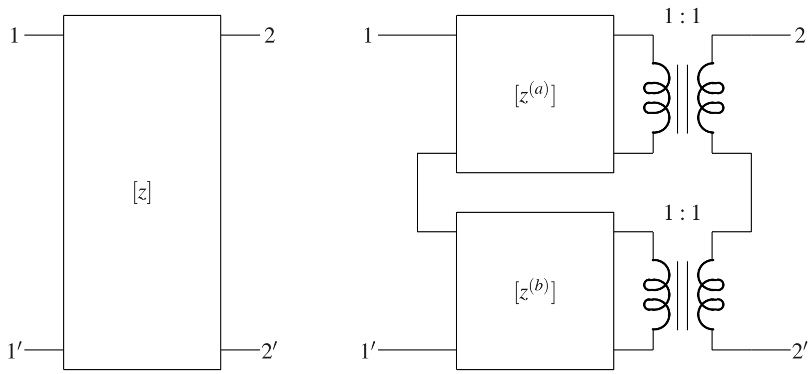


Fig. 9.13 A two-port is decomposed into two 2-ports connected in series

2. Let $[z]$ be the open-circuit impedance matrix of a 2-port. Let an ideal transformer of turns ratio $1 : n$ be connected across port-2. Then the augmented 2-port has an impedance matrix $[z']$ whose elements are:

$$z'_{11} = z_{11}\,, \qquad z'_{12} = nz_{12}\,, \qquad z'_{22} = n^2 z_{22}\,. \tag{9.113}$$

3. A "Tee" of inductors with inductances L_a, L_b and L_c that satisfy the relation $L_aL_b + L_aL_c + L_bL_c = 0$, shown in Fig. 9.14, is equivalent to a 2-port of two coupled inductors with unity coefficient of coupling, which in turn is equivalent to a 2-port consisting of an inductor of inductance $L_a + L_b$ connected across the primary of an ideal transformer of turns ratio $1:n$, where $n = L_b/(L_a + L_b)$.

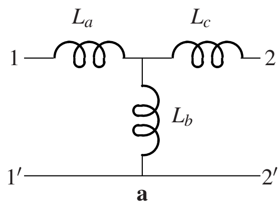


Fig. 9.14 The three 2-ports are equivalent, provided the following relations hold: $L_aL_b + L_aL_c + L_bL_c = 0$; $L_{11} = L_a + L_b$; $M = L_b$; and $n = L_b/(L_a + L_b)$

4. A "Tee" of capacitors with reciprocal capacitances Γ_a, Γ_b, and Γ_c that satisfy the relation $\Gamma_a\Gamma_b + \Gamma_a\Gamma_c + \Gamma_b\Gamma_c = 0$, shown in Fig. 9.15, is equivalent to a 2-port consisting of a capacitor of reciprocal capacitance $\Gamma_a + \Gamma_b$ connected across the primary of an ideal transformer of turns ratio $1:n$, where $n = \Gamma_b/(\Gamma_a + \Gamma_b)$.

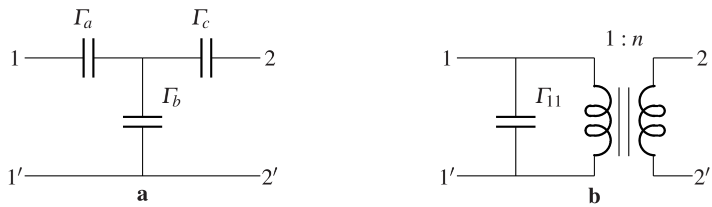


Fig. 9.15 The two 2-ports are equivalent, provided the following relations hold: $\Gamma_a\Gamma_b + \Gamma_a\Gamma_c + \Gamma_b\Gamma_c = 0$; $\Gamma_{11} = \Gamma_a + \Gamma_b$; and $n = \Gamma_b/(\Gamma_a + \Gamma_b)$

9.7.2 Synthesis procedure

It is best to do an example. Assume the input impedance function associated with a transmission function that we wish to realize is:[4]

$$Z(s) = \frac{6s^2 + 5s + 6}{2s^2 + 4s + 4}. \tag{9.114}$$

The even polynomial $E(s^2)$ is:

[4] Taken from [72], but no derivation is given in the text.

$$E(s^2) = m_1m_2 - n_1n_2 = 12s^4 + 16s^2 + 24\,, \tag{9.115}$$

which is not a perfect square. The roots, which are the transmission zeros, are:

$$\begin{aligned} s_o &= \sigma_o + j\omega_o = 0.61137 + j1.02002\,, & s_o^* &= 0.61137 - j1.02002\,, \\ -s_o &= -0.61137 + j1.02002\,, & -s_o^* &= -0.61137 - j1.02002\,. \end{aligned}$$

So we augment $Z(s)$ with the polynomial $(s+\sigma_o+j\omega_o)(s+\sigma_o-j\omega_o)$ to get

$$Z(s) = \frac{6s^4 + 12.33644s^3 + 20.59898s^2 + 14.40751s + 8.48528}{2s^4 + 6.44548s^3 + 11.71939s^2 + 10.54815s + 5.65685}\,. \tag{9.116}$$

We make the identification

$$z_{11}(s) = \frac{6s^4 + 20.59898s^2 + 8.48528}{6.44548s^3 + 10.54782s}\,. \tag{9.117}$$

In this case, we remove an inductor in series with a capacitor from $z_{11}(s)$ to get a remainder $z_1(s)$ which will have zeros at the transmission zeros:

$$z_1(\sigma_o + j\omega_o) = z_{11}(\sigma_o + j\omega_o) - (\sigma_o + j\omega_o)L_a - \frac{\Gamma_a}{(\sigma_o + j\omega_o)} = 0\,. \tag{9.118}$$

Solving for L_a and the reciprocal capacitance Γ_a, we get

$$L_a = 1.5\,, \qquad \Gamma_a = 1.5\,, \tag{9.119}$$

and the remainder impedance is

$$z_1(s) = z_{11}(s) - L_a s - \frac{\Gamma_a}{s} = \frac{-0.569115(s^4 + \frac{4}{3} + 2)}{s(s^2 + 1.636468)}\,. \tag{9.120}$$

We next reciprocate $z_1(s)$ to get a shunt admittance $y_1(s)$ to realize the transmission zeros. In this case, the realization of the transmission zeros requires all of $y_1(s)$ with no remainder. The shunt admittance is obtained by expanding $z_1(s)$ in partial fractions:

$$z_1(s) = -0.569115s - \frac{0.6955}{s} + \frac{0.8680s}{s^2 + 1.63648}\,, \tag{9.121}$$

which is a series combination of a negative inductor, a negative capacitor and a parallel LC impedance. The element values are:

$$\begin{aligned} L_b &= -0.569115\,, & \Gamma_b &= -0.6955\,, \\ C &= \frac{1}{0.8680} = 1.15207\,, & L &= \frac{1}{1.63648C} = 0.5304\,. \end{aligned}$$

The removal of L_a and Γ_a introduces a pole at ∞ and at zero which must be canceled by adding an impedance $L_c s + \Gamma_c/s$ in series with port-2. The element values are found from:

$$L_c = \frac{-L_a L_b}{L_a + L_b} = 0.91705\,, \tag{9.122}$$

$$\Gamma_c = \frac{-\Gamma_a \Gamma_b}{\Gamma_a + \Gamma_b} = 1.29677\,. \tag{9.123}$$

We will also need the turns ratio of an ideal transformer:

$$n = \frac{\Gamma_b}{\Gamma_a + \Gamma_b} = -0.86451\,. \tag{9.124}$$

The lossless 2-port so far has a configuration shown in Fig. 9.16**a**. The $1/1$ transformers can be removed by replacing the "Tee" of inductors by coupled inductors as shown in Fig. 9.14, and by replacing the "Tee" of capacitors by its equivalent shown in Fig. 9.15. The circuit is unchanged if we insert a transformer with its primary side connected to port-2 of turn ratio $n:1$, with the load resistance multiplied by $1/n^2$ as shown in Fig. 9.16**b**. Now all the ideal transformers can be eliminated if L_{22} is divided by n^2, the mutual inductance M is divided by n, and the combination of L and the ideal transformer is replaced by coupled inductors as in Fig. 9.14. The final circuit is shown in Fig. 9.17, where the element values are:

$$\begin{array}{llll} L_1 = 0.93088\,, & L_2 = 0.46554\,, & M_1 = 0.65831\,, & C_1 = 1.15207\,, \\ L_3 = 0.53044\,, & M_4 = 0.70968\,, & M_2 = -0.61349\,, & C_2 = 1.243\,. \end{array}$$

in H and F. Port-2 is terminated in a resistance of 1.338 Ω .

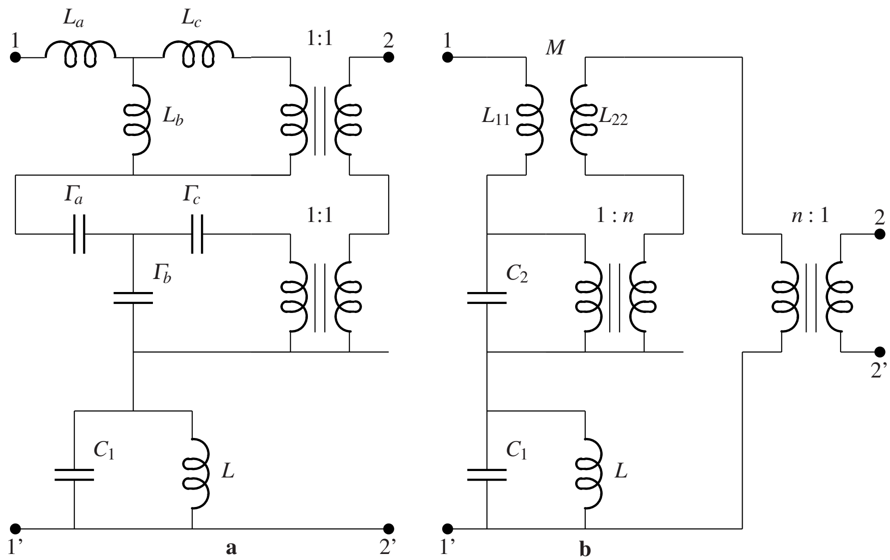


Fig. 9.16 Preliminary Darlington D-section **a** and its intermediate equivalent **b**

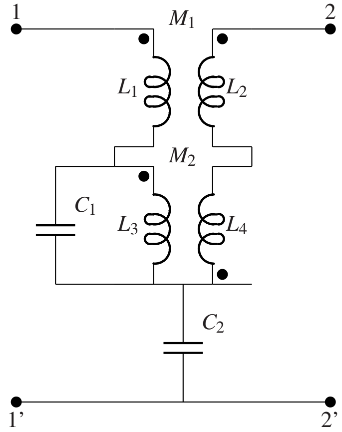


Fig. 9.17 Darlington D-section to realize complex transmission zeros

We have presented a procedure to synthesize an impedance function whose even polynomial $E(s^2) = m_1 m_2 - n_1 n_2$ has double complex zeros with quadrantal symmetry. We have not proved that the procedure will always lead to a realizable circuit. In other words, we need to prove that the elements we obtain in the process are realizable. For example, we need to show that L_a and Γ_a are always positive, that C_2 and L are positive, and that the remainder impedance after removal of one D-section is still a positive real function. These details and others are found in [15].

9.8 Remarks

Darlington's synthesis of transfer functions is the most general of its kind in that any transmission function $S_{21}(s)$ which is analytic in the right-half s-plane and whose magnitude squared is less than unity can be realized as the transmission function of a lossless two-port terminated in resistance at the input and output ports. The circuit realization requires the use of perfect coupled inductors and is therefore not practical. Countless attempts have been made to find a general synthesis procedure which will realize a general transfer function without the use of perfect coupled inductors or ideal transformers. So far, we have not been able to find any. In fact, we do not even know the necessary and sufficient conditions for a rational function with zeros all on the $j\omega$-axis to be realizable as the transfer function of a lossless *ladder* terminated in resistance at both ends without the use of transformers. The synthesis given in Sect. 9.4 is simply a demonstration of how we find a realization if one exists.

Problems

9.1. Find a lossless ladder with minimum number of inductors, terminated in a one-ohm resistor at both ends, that realizes each of following transmission power gain functions. Check your answer by computing the frequency response of the circuit from node analysis.

$$(a)\ |S_{21}(j\omega)|^2 = \frac{1}{1+\omega^6}.$$
$$(b)\ |S_{21}(j\omega)|^2 = \frac{\omega^2}{1+\omega^6}.$$
$$(c)\ |S_{21}(j\omega)|^2 = \frac{\omega^4}{1+\omega^6}.$$
$$(d)\ |S_{21}(j\omega)|^2 = \frac{\omega^6}{1+\omega^6}.$$

9.2. Find a lossless ladder terminated in a one-ohm resistor at both ends that realizes each of the following transmission gain functions. There are many possible realizations. Choose one in each case that has the fewest inductors.

$$(a)\ |S_{21}(j\omega)|^2 = \frac{4\omega^2}{4\omega^6+9\omega^4+6\omega^2+1}.$$
$$(b)\ |S_{21}(j\omega)|^2 = \frac{16\omega^4}{4\omega^6+9\omega^4+6\omega^2+1}.$$
$$(c)\ |S_{21}(j\omega)|^2 = \frac{4\omega^6}{4\omega^6+9\omega^4+6\omega^2+1}.$$

9.3. Find a lossless ladder with minimum number of inductors, terminated in one-ohm at both ends, that realizes the following transmission power gain. What should the constant K be? Check your answer by computing the frequency response of the circuit from node analysis.

$$|S_{21}(j\omega)|^2 = \frac{K}{1+\omega^{10}}.$$

9.4. Find a lossless ladder with minimum number of inductors, terminated in 1 Ω at port-1 and 4 Ω at port-2, that realizes the following high-pass transmission function. What should the constant K be? Plot the frequency response.

$$|S_{21}(j\omega)|^2 = \frac{K\omega^{10}}{1+\omega^{10}}.$$

9.5. Find a lossless two-port with termination of 3 Ω at port-1 and 1 Ω at port-2 that realizes the following transfer function in magnitude squared:

$$|H(j\omega)|^2 = \frac{K}{40.96\omega^{10} - 102.4\omega^8 + 89.6\omega^6 - 32\omega^4 + 4\omega^2 + 1}.$$

9.6. Realize the following function as the transmission function of a lossless 2-port terminated in one-ohm at both ends.

$$|S_{21}(j\omega)|^2 = \frac{9(\omega^2+1)^2}{25\omega^4 + 34\omega^2 + 9}.$$

9.7. Find a lossless ladder inserted between terminations of equal resistance of 1 Ω such that the transmission power gain is each of the following. Use as few inductors as possible. Plot the frequency response of the circuit.

$$(a)\ |S_{21}(j\omega)|^2 = \frac{16\omega^4 - 128\omega^2 + 256}{25\omega^6 - 60\omega^4 - 64\omega^2 + 256}.$$
$$(b)\ |S_{21}(j\omega)|^2 = \frac{64\omega^4 - 32\omega^2 + 4}{81\omega^6 + 46\omega^4 - 31\omega^2 + 4}.$$
$$(c)\ |S_{21}(j\omega)|^2 = \frac{16\omega^4 - 64\omega^2 + 64}{36\omega^6 + 25\omega^4 - 64\omega^2 + 64}.$$
$$(d)\ |S_{21}(j\omega)|^2 = \frac{144\omega^4 - 288\omega^2 + 144}{121\omega^6 + 202\omega^4 - 287\omega^2 + 144}.$$

9.8. Find a lossless ladder inserted between a 1 Ω resistor at the input end and a 4 Ω resistor at the output end such that the transmission power gain is each of the following. Use as few inductors as possible. Plot the frequency response of the circuit.

$$(a)\ |S_{21}(j\omega)|^2 = \frac{64\omega^4 - 128\omega^2 + 64}{324\omega^6 - 179\omega^4 - 155\omega^2 + 100}.$$
$$(b)\ |S_{21}(j\omega)|^2 = \frac{4\omega^4 - 16\omega^2 + 16}{25\omega^6 - 21\omega^4 - 21\omega^2 + 25}.$$
$$(c)\ |S_{21}(j\omega)|^2 = \frac{64\omega^4 - 128\omega^2 + 64}{400\omega^6 - 455\omega^4 - 11\omega^2 + 100}.$$
$$(d)\ |S_{21}(j\omega)|^2 = \frac{256\omega^4 - 512\omega^2 + 256}{784\omega^6 - 615\omega^4 - 484\omega^2 + 400}.$$

9.9. Find a lossless two-port terminated in 1 Ω at port-1 and 4 Ω at port-2 that realizes the following transmission power gain. Use as few inductors as possible.

$$|S_{21}(j\omega)|^2 = \frac{a_8\omega^8 + a_6\omega^6 + a_4\omega^4 + a_2\omega^2 + a_0}{b_{10}\omega^{10} + b_8\omega^8 + b_6\omega^6 + b_4\omega^4 + b_2\omega^2 + b_0}.$$

$$a_8 = 0.64\,, \qquad a_6 = -7.10819276\,, \qquad a_4 = 28.38873557\,,$$
$$a_2 = -48.04615686\,, \qquad a_0 = 29.24010253\,, \qquad b_{10} = 97.61883993\,,$$
$$b_8 = -270.08164395\,, \qquad b_6 = 259.58469411\,, \qquad b_4 = -70.18705334\,,$$
$$b_2 = -57.64275595\,, \qquad b_0 = 45.68766020\,.$$

Plot the transmission power gain and notice the frequency response is that of an elliptic low-pass filter. Check your answer by computing the same from circuit analysis.

9.10. Find a lossless ladder inserted between a $1\,\Omega$ resistor at the input end and a $4\,\Omega$ resistor at the output such that the transmission power gain is each of the following. Use as few inductors as possible. Plot the frequency response of the resultant circuit.

$$(a)\ |S_{21}(j\omega)|^2 = \frac{4(4\omega^8 - 40\omega^6 + 132\omega^4 - 160\omega^2 + 64)}{324\omega^{10} - 1379\omega^8 + 1937\omega^6 - 720\omega^4 - 384\omega^2 + 256}\,.$$
$$(b)\ |S_{21}(j\omega)|^2 = \frac{4(64\omega^8 - 640\omega^6 + 2112\omega^4 - 2560\omega^2 + 1024)}{1849\omega^{10} - 6264\omega^8 + 3472\omega^6 + 9472\omega^4 - 10240\omega^2 + 4096}\,.$$
$$(c)\ |S_{21}(j\omega)|^2 = \frac{4(36\omega^8 - 216\omega^6 + 468\omega^4 - 432\omega^2 + 144)}{2500\omega^{10} - 7731\omega^8 + 6637\omega^6 - 144\omega^4 - 1532\omega^2 + 576}\,.$$
$$(d)\ |S_{21}(j\omega)|^2 = \frac{4(9\omega^8 - 54\omega^6 + 117\omega^4 - 108\omega^2 + 36)}{441\omega^{10} - 1391\omega^8 + 1404\omega^6 - 279\omega^4 - 311\omega^2 + 144}\,.$$

9.11. Synthesize each of the following transmission power gain functions in a Darlington C-section inserted between two 1Ω resistors at the input and output.

$$(a)\ |S_{21}(j\omega)|^2 = \frac{4(36\omega^4 + 36\omega^2 + 9)}{169\omega^4 + 244\omega^2 + 36}\,.$$
$$(b)\ |S_{21}(j\omega)|^2 = \frac{4(16\omega^4 + 16\omega^2 + 4)}{289\omega^4 + 89\omega^2 + 16}\,.$$

9.12. Synthesize each of the following transmission power gain functions in a Darlington C-section inserted between a 1Ω resistor at the input and a $4\,\Omega$ resistor at the output.

$$(a)\ |S_{21}(j\omega)|^2 = \frac{144\omega^4 + 288\omega^2 + 144}{1369\omega^4 - 326\omega^2 + 225}\,.$$
$$(b)\ |S_{21}(j\omega)|^2 = \frac{64\omega^4 + 64\omega^2 + 16}{100\omega^4 + 125\omega^2 + 25}\,.$$

Chapter 10
Filter Design

Filter design begins with a specification of the frequency characteristics of a circuit. From this specification, we attempt to find a rational function whose frequency response approximates the desired characteristics. The rational function must be realizable as the transfer function of an *RLC* circuit, preferably a lossless 2-port terminated in resistors with specified resistances. Once a realizable transfer function has been found, we apply the synthesis techniques of the last chapter to realize the filter. In this chapter we learn to design Butterworth, Chebyshev and Cauer filters, and to demonstrate why designs based on the realization of transmission power gains lead to filters whose loss sensitivity in the passband is zero at zero loss frequencies.

10.1 Filter functions

We will start with low pass filters. High-pass, band-pass and band-elimination filters can be derived from low-pass filters by frequency transformation, as we will see later. Our problem is to find coefficients a_k and b_k in the following transmission power gain:

$$|S_{21}(j\omega)|^2 = A_0 \frac{1 + a_1\omega^2 + a_2\omega^4 + \cdots + a_{n-1}\omega^{2(n-1)}}{1 + b_1\omega^2 + b_2\omega^4 + \cdots + b_n\omega^{2n}} \tag{10.1}$$

such that

1. $|S_{21}(j\omega)|^2$ approximates a constant A_0 for $0 \le \omega \le \omega_p$;
2. $|S_{21}(j\omega)|^2$ approximates another constant $A_s \ll A_0$, preferably zero, for $\omega_s \le \omega \le \infty$;
3. $|S_{21}(j\omega)|^2$ is realizable as the transmission power gain of a lossless 2-port terminated in R_1 at port-1 and R_2 at port-2;
4. The 2-port should have as few inductors as possible.

In Eq. (10.1), A_0 is the power gain at DC, and usually $a_n = 0$ because in a low-pass filter, we want $S_{21}(j\infty) = 0$. The frequency range $[0, \omega_p]$ is called the *passband* and

ω_p is the *passband edge*. The range $[\omega_s, \infty]$ is called the *stop-band* and ω_s is the *stop-band edge*. The band $[\omega_p, \omega_s]$ is called the *transition band*. Fig. 10.1 shows a sketch of the desired frequency characteristics.

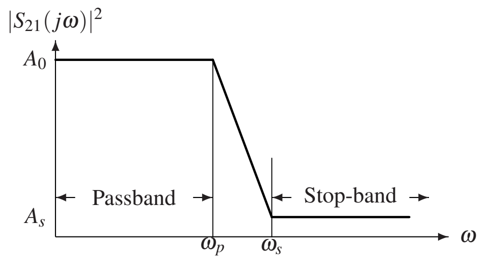


Fig. 10.1 A sketch of the frequency characteristics of a low-pass filter

We will study three types of approximation:

1. Butterworth or maximally flat low pass characteristic;
2. Chebyshev or equi-ripple low pass characteristic; and
3. Cauer or elliptic low pass characteristic, which have equal ripples in both the passband and stop-band.

It must be emphasized that there are many other types, (see [18, 56, 72]) but these three are the most common in practice.

10.2 Maximally flat approximation

This is the simplest approximation. The transmission power gain is a smooth, monotonically decreasing function of frequency from DC to infinite frequency. We first obtain an expression of the transmission power gain and then derive a design formula of a low-pass filter with this characteristic.

10.2.1 Transmission power gain

In this approximation, we require $|S_{21}(j\omega)|^2$ to approximate the DC gain A_0 in a manner that its Taylor series expansion about $\omega = 0$ takes the form:

$$|S_{21}(j\omega)|^2 = A_0 + c_n\omega^{2n} + c_{n+1}\omega^{2n+2} + \cdot, \tag{10.2}$$

namely, the first $2n-1$ derivatives of $|S_{21}(j\omega)|^2$ at $\omega = 0$ are zero. Writing

$$|S_{21}(j\omega)|^2 - A_0 = A_0 \frac{(a_1-b_1)\omega^2 + (a_2-b_2)\omega^4 + \cdots + (a_{n-1}-b_{n-1})\omega^{2n-2} - b_n\omega^{2n}}{1+b_1\omega^2+\cdots+b_n\omega^{2n}}, \tag{10.3}$$

we see this requirement is satisfied if

$$a_k = b_k, \quad k = 1,\ldots,n-1. \tag{10.4}$$

Second, for ω near infinity, we write

$$|S_{21}(j\omega)|^2 = A_0 \frac{a_{n-1}\omega^{-2} + a_{n-2}\omega^{-4} + \cdots + \omega^{-2n}}{b_n + b_{n-1}\omega^{-2} + \cdots + \omega^{-2n}}, \tag{10.5}$$

and require the Taylor series about $\omega = \infty$ to take the form:

$$|S_{21}(j\omega)|^2 = d_n\omega^{-2n} + d_{n+1}\omega^{-(2n+2)} + \cdots, \tag{10.6}$$

namely it goes to zeros as the $2n$th power of ω. To meet this requirement, we must have

$$a_k = 0 \quad \text{for} \quad k = 1,\ldots,n-1. \tag{10.7}$$

The transmission power gain becomes:

$$|S_{21}(j\omega)|^2 = \frac{A_0}{1+b_n\omega^{2n}}. \tag{10.8}$$

Let us define the passband edge ω_p as the frequency at which the power gain deviates from its DC value by a small amount, specified as:

$$|S_{21}(j\omega_p)|^2 = \frac{A_0}{1+b_n\omega_p^{2n}} = \frac{A_0}{1+\varepsilon^2}, \tag{10.9}$$

where ε is a small constant. It follows that a maximally flat transmission power gain is:

$$|S_{21}(j\omega)|^2 = \frac{A_0}{1+\varepsilon^2\left(\dfrac{\omega}{\omega_p}\right)^{2n}}. \tag{10.10}$$

It is convenient to define a normalized frequency:

$$p = \frac{s}{\omega_p} = \frac{\sigma}{\omega_p} + j\frac{\omega}{\omega_p} = x + jy, \tag{10.11}$$

and the transmission power gain function becomes:

$$|S_{21}(jy)|^2 = \frac{A_0}{1+\varepsilon^2 y^{2n}}, \tag{10.12}$$

which has the characteristics shown in Fig. 10.2 for various values of n, called the order of approximation. The parameter ε^2 is a measure of the maximum deviation from a constant value in the passband.

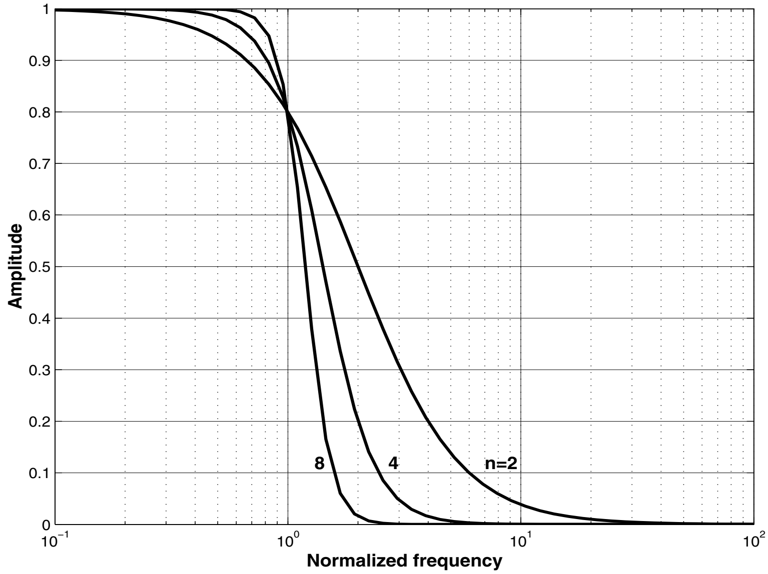


Fig. 10.2 Maximally flat characteristics for $\varepsilon^2 = 0.25$.

10.2.2 Transmission poles and zeros

Since the numerator of $|S_{21}(j\omega)|^2$ is a constant, all the transmission zeros are at infinity. To find the poles, we write the function as follows.

$$|S_{21}(jy)|^2 = S_{21}(p)S_{21}(-p)|_{p=jy} = \left.\frac{A_0}{1+\varepsilon^2(-p^2)^n}\right|_{p=jy} . \tag{10.13}$$

The poles are found from

$$\varepsilon^2(-p^2)^n = -1 = e^{j(2k-1)\pi}, \tag{10.14}$$

and are given by

$$p_k = \zeta^{1/n}(-\sin\theta_k + j\cos\theta_k), \quad \text{for} \quad k = 1,2,\ldots,2n, \tag{10.15}$$

where

$$\zeta = \frac{1}{\varepsilon}, \qquad \theta_k = \frac{(2k-1)\pi}{2n}.$$

Since

$$|p_k|^2 = \zeta^{2/n}(\sin^2\theta_k + \cos^2\theta_k) = \zeta^{2/n} \tag{10.16}$$

independent of k, the poles lie on a circle of radius $\zeta^{1/n}$ in the s-plane. To be realizable, $S_{21}(p)$ must have poles in the left-half of the s-plane, and its expression is

$$S_{21}(p) = \frac{\sqrt{A_0}}{\prod_{k=1}^{n/2}(p^2 + 2\zeta^{1/n}(\sin\theta_k)p + \zeta^{2/n})}, \quad n\text{=even}; \tag{10.17}$$

$$S_{21}(p) = \frac{\sqrt{A_0}}{(p+\zeta^{1/n})\prod_{k=1}^{(n-1)/2}(p^2 + 2\zeta^{1/n}(\sin\theta_k)p + \zeta^{2/n})}, \quad n\text{=odd}. \tag{10.18}$$

Once we have $S_{21}(p)$, we can find the reflection coefficient function $S_{11}(p)$ and input impedance $Z(p)$ and proceed to synthesize the 2-port terminated in resistances at the ports. Since all the transmission zeros are at infinity, the 2-port can be an *LC* ladder of the form studied in the last chapter. If port-1 is terminated in R_1 and port-2 in R_2, then the *DC* power gain A_0 is:

$$A_0 = |S_{21}(0)|^2 = 4\frac{R_1}{R_2}\left|\frac{V_2}{E}\right|^2_{\omega=0} = \frac{4R_1R_2}{(R_1+R_2)^2}. \tag{10.19}$$

10.2.3 Design considerations

In practice, the filter specifications are usually expressed in dB. Let

$$\alpha(y) = -10\log|S_{21}(jy)|^2 = -10\log A_0 + 10\log(1+\varepsilon^2 y^{2n}). \tag{10.20}$$

We call $\alpha(y)$ the *loss function*. The loss at DC is:

$$\alpha_0 = \alpha(0) = -10\log A_0, \tag{10.21}$$

and the difference:

$$\alpha(y) - \alpha_0 = 10\log(1+\varepsilon^2 y^{2n}) \tag{10.22}$$

is called the *insertion loss* of the filter. Making use of Eqs. (10.19) and (10.22), we have:

$$\alpha(y) - \alpha_0 = -10\log\left(\frac{1}{2R_2}\left|\frac{V(jy)}{E}\right|^2\right) + 10\log\left(\frac{1}{2R_2}\left|\frac{V(0)}{E}\right|^2\right). \tag{10.23}$$

The first term is the power delivered to the load R_2 with the filter inserted between the load and source (with internal resistance R_1), and the second term is the power in R_2 with R_2 directly connected to the source. The difference in dB is the insertion loss.

The specifications are:

1. In the passband, the insertion loss may not be more than α_{max} dB, namely

$$\alpha(y) - \alpha_0 \le \alpha_{max} \quad \text{for} \quad y \le y_p\,, \tag{10.24}$$

where y_p is the (normalized) passband edge. Since the maximally flat transmission function is monotonically decreasing, this condition can be met if we choose ε such that

$$10\log(1+\varepsilon^2 y_p^{2n}) = \alpha_{max} \tag{10.25}$$

or,

$$\varepsilon^2 = 10^{0.1\alpha_{max}} - 1 \tag{10.26}$$

2. In the stop-band, the insertion loss may not be less than α_{min} dB, i.e.

$$\alpha(y) - \alpha_0 \ge \alpha_{min} \quad \text{for} \quad y \ge y_s\,, \tag{10.27}$$

where y_s is the (normalized) stop-band edge. This condition implies that

$$10\log(1+\varepsilon^2 y_s^{2n}) = \alpha_{min}\,. \tag{10.28}$$

Combining Eqs. (10.26) and (10.28), we obtain a design formula:

$$n = \left\lceil \frac{\log \dfrac{10^{0.1\alpha_{min}} - 1}{10^{0.1\alpha_{max}} - 1}}{2\log \dfrac{\omega_s}{\omega_p}} \right\rceil\,. \tag{10.29}$$

As an example, the following specifications:

1. Passband insertion loss is not more than 0.25 dB;
2. Stop-band insertion loss is not less than 40 dB;
3. Passband edge is $f_p = 80$ KHz; and
4. Stop-band edge is $f_s = 300$ KHz.

will require that $\varepsilon^2 = 0.0592537$ by Eq. (10.26) and $n = 5$ by Eq. (10.29).

10.2.4 Filter synthesis

Let us find a lossless ladder terminated in 1000 Ω at port-1 and 4000 Ω at port-2 that realizes the 5th order maximally flat filter of this example.

We first normalize the impedance with respect to R_1, so $R_1 = 1$ and $R_2 = 4\ \Omega$. The DC power gain is $A_0 = 0.64$. The transmission gain function $S_{21}(p)$ satisfies

$$S_{21}(p)S_{21}(-p) = \frac{0.64}{1-\varepsilon^2 p^{10}}\,, \tag{10.30}$$

and the reflection function satisfies

$$S_{11}(p)S_{11}(-p) = 1 - S_{21}(p)S_{21}(-p) = \frac{0.36-\varepsilon^2 p^{10}}{1-\varepsilon^2 p^{10}}\,. \tag{10.31}$$

Following the synthesis procedure we studied in the last chapter, assigning all the right half plane zeros to $S_{11}(p)$, and noting that $R_2 > R_1$, we find the input impedance to be:

$$Z(p) = \frac{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}{b_5p^5 + b_4p^4 + b_3p^3 + b_2p^2 + b_1s + b_0}\,, \tag{10.32}$$

where

$$\begin{aligned} &a_4 = 8.168787\,, && a_3 = 1.702865\,, && a_2 = 21.22002\,, && a_1 = 3.361804\,,\\ &a_0 = 6.572978\,, && b_5 = 2.000000\,, && b_4 = 0.416920\,, && b_3 = 16.72573\,,\\ &b_2 = 3.226699\,, && b_1 = 16.68113\,, && b_0 = 1.643244\,. \end{aligned}$$

Expanding m_1/n_2 in continued fraction about $p = \infty$, we obtain a realization shown in Fig. 10.3 where the element values are given below.

Element	p-domain	s-domain	1000-Ω basis
C_1	0.244834	0.487082 µF	0.487082 nF
L_1	0.708460	1.409437 µH	1.409437 mH
C_2	1.093730	2.175906 µF	2.175906 nF
L_2	1.337372	2.660618 µH	2.660618 mH
C_3	1.199270	2.385873 µF	2.385873 nF

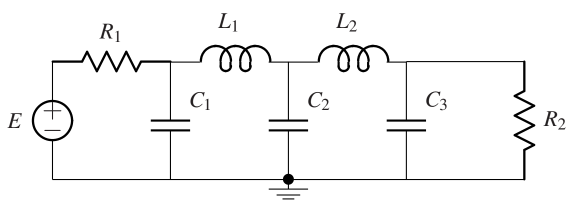


Fig. 10.3 A maximally flat lowpass filter of order 5 with $R_1 = 1000\,\Omega$ and $R_2 = 4000\,\Omega$

The frequency response as computed by node analysis is shown in Fig. (10.4). It is seen that the insertion loss meets the filter specifications.

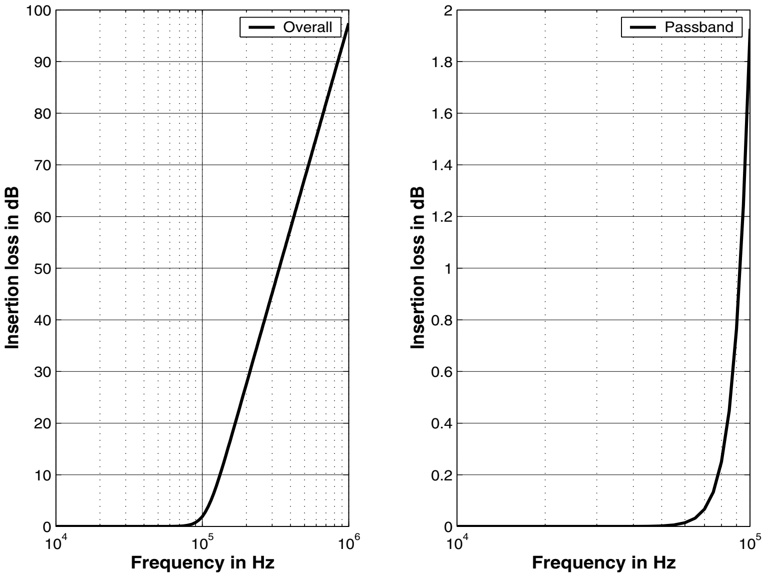


Fig. 10.4 Insertion loss of a fifth order Butterworth low-pass filter

10.3 Chebyshev Filters

As we did in the Butterworth case, let the rational function $|S_{21}(j\omega)|^2$ be written as

$$|S_{21}(j\omega)|^2 = A\frac{1+a_1\omega^2+\cdots+a_{n-1}\omega^{2n-2}}{1+b_1\omega^2+\cdots+b_{n-1}\omega^{2n-2}+b_n\omega^{2n}}\,. \tag{10.33}$$

We choose a_k's and b_k's such that

1. $|S_{21}(j\omega)|^2$ approximates a constant in the passband in the sense that the maximum deviation is minimized; and
2. Its first $2n-1$ derivatives are zero as ω approaches ∞ in the stopband.

The second requirement suggests that

$$a_1 = a_2 = \cdots = a_{n-1} = 0\,.$$

So we can write $|S_{21}(jy)|^2$ as

$$|S_{21}(jy)|^2 = \frac{A}{1+\varepsilon^2 T_n^2(y)}, \tag{10.34}$$

where $T_n(y)$ is an n^{th} order polynomial to be determined, and y is the normalized frequency

$$y = \omega/\omega_p$$

where ω_p is the passband edge, and ε is a parameter of small value.

Requirement 1 implies that $|S_{21}(jy)|^2$ will oscillate about a constant with all the maxima having the same value and all the minima being equal. It follows that $|S_{21}(jy)|^2$ will have a form shown in Fig. 10.5. All the maxima have a value of one and all the minima are equal to $1/(1+\varepsilon^2)$. At the minima, we have set the value of

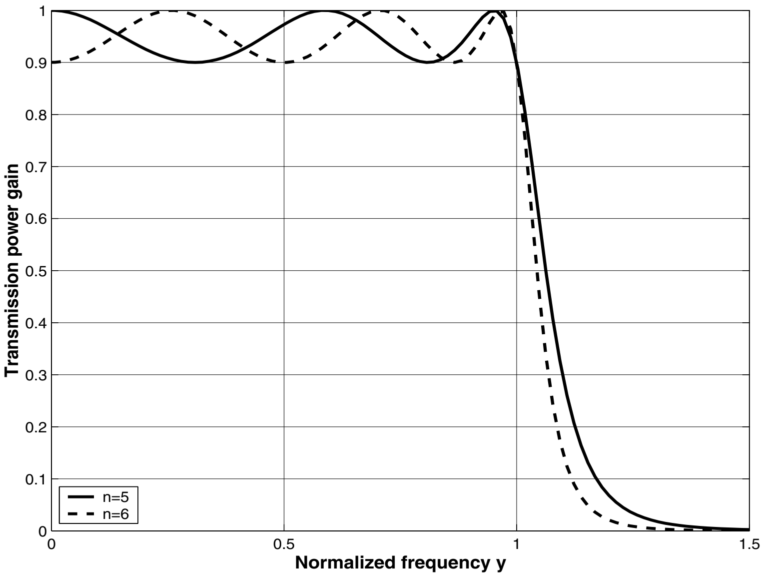


Fig. 10.5 Typical Chebyshev low-pass filter response

$T_n^2(y)$ to one for convenience. At the maxima, $T_n^2(y) = 0$. Therefore, we expect $T_n(y)$ will have a form shown in Fig. 10.6. From Fig. 10.6, we can deduce the properties of the polynomial $T_n(y)$, as follows.

1. The maxima of $T_n(y)$ are all equal to 1.
2. The minima of $T_n(y)$ are all equal to -1.
3. All the zeros of $T_n(y)$ lie between $y = -1$ and $y = 1$.
4. $T_n(1) = 1$.
5. $T_n(y)$ is odd for odd n, and even for even n.

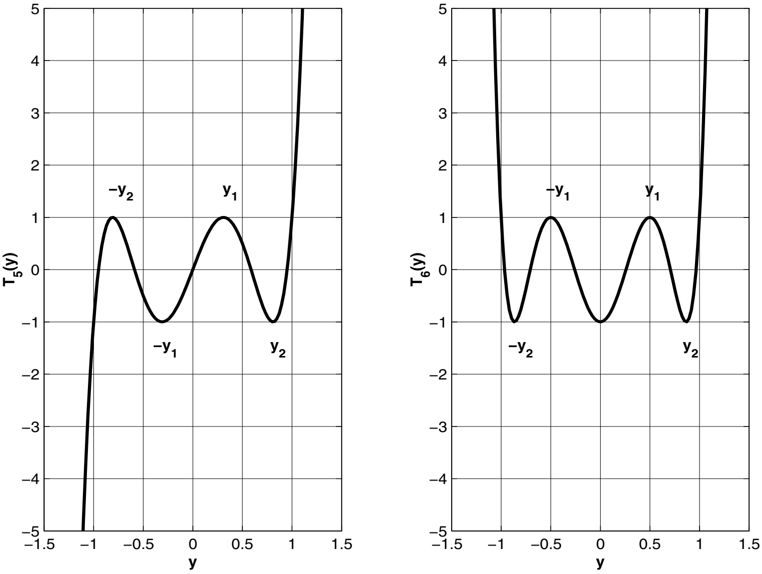


Fig. 10.6 Expected form of polynomials $T_5(y)$ and $T_6(y)$

10.3.1 Derivation of $T_n(y)$

We now proceed to derive an explicit expression of the polynomial $T_n(y)$. In addition to the properties listed in the last section, we note that at the maxima and minima, the derivatives of $T_n(y)$ are zero. Hence, with reference to Fig. 10.6, we can write

$$\frac{\mathrm{d}T_n}{\mathrm{d}y} = C_1(y-y_1)(y+y_1)(y-y_2)(y+y_2)\cdots, \tag{10.35}$$

where $\pm y_k$ are the values of y at which the maxima and minima occur. Thus we have

$$\frac{\mathrm{d}T_n}{\mathrm{d}y} = c_1 \prod_{k=1}^{(n-1)/2} (y^2 - y_k^2) \qquad n = odd\,; \tag{10.36}$$

$$\frac{\mathrm{d}T_n}{\mathrm{d}y} = c_1 y \prod_{k=1}^{n/2} (y^2 - y_k^2) \qquad n = even\,. \tag{10.37}$$

Next consider $T_5(y)+1$. This function has a double zero at $-y_1$ and y_2 and a simple zero at -1. Similarly, the function $T_5(y)-1$ has a double zero at y_1 and $-y_2$ and a simple zero at 1. We can therefore write

$$T_5(y)+1 = c_2(y+1)(y+y_1)^2(y-y_2)^2\,, \tag{10.38}$$

$$T_5(y)-1 = c_3(y-1)(y-y_1)^2(y+y_2)^2\,, \tag{10.39}$$

or

$$T_5^2(y) - 1 = c_4(y^2 - 1)(y^2 - y_1^2)^2(y^2 - y_2^2)^2 .$$

In general, we have

$$T_n^2(y) - 1 = c_5(y^2 - 1) \prod_{k=1}^{(n-1)/2} (y^2 - y_k^2)^2 \qquad n = odd , \tag{10.40}$$

$$T_n^2(y) - 1 = c_5 y^2(y^2 - 1) \prod_{k=1}^{n/2} (y^2 - y_k^2)^2 \qquad n = even . \tag{10.41}$$

Comparing Eq. (10.36) with Eq. (10.40), and Eq. (10.37) with Eq. (10.41), we get

$$\left(\frac{\mathrm{d}T_n}{\mathrm{d}y}\right)^2 = M^2 \frac{1 - T_n^2}{1 - y^2}$$

or

$$\frac{\mathrm{d}T_n}{\sqrt{1 - T_n^2}} = M \frac{\mathrm{d}y}{\sqrt{1 - y^2}} , \tag{10.42}$$

whose solution is

$$T_n(y) = \cos\left(M \cos^{-1} y + C\right) \tag{10.43}$$

As y varies from -1 to 1, $T_n(y)$ goes through zeros n times and $M \cos^{-1} y$ must go through $n\pi$ radians. It follows that $M = n$. Since we require that $T_n(1) = 1$, $C = 0$. We now have the explicit expression for $T_n(y)$ given by

$$T_n(y) = \cos(n \cos^{-1} y) . \tag{10.44}$$

That $T_n(y)$ is a polynomial can be seen from the following:

$$T_0(y) = 1 , \tag{10.45}$$

$$T_1(y) = y , \tag{10.46}$$

$$T_{n+1}(y) = 2yT_n(y) - T_{n-1} , \tag{10.47}$$

from which we deduce

$$\begin{aligned}
T_2(y) &= 2y^2 - 1 , \\
T_3(y) &= 4y^3 - 3y , \\
T_4(y) &= 8y^4 - 8y^2 + 1 , \\
T_5(y) &= 16y^5 - 20y^3 + 5y , \\
T_6(y) &= 32y^6 - 48y^4 + 18y^2 - 1 , \\
&\cdots .
\end{aligned}$$

The recursion relation of Eq. (10.47) can be derived by considering the sum of $\cos(n+1)\theta$ and $\cos(n-1)\theta$, where $\cos\theta = y$.

10.3.2 $S_{21}(p)$ and $S_{11}(p)$

Returning to $S_{21}(j\omega)$, we now have from Eq. (10.34)

$$S_{21}(p)S_{21}(-p) = \frac{A}{1+\varepsilon^2 T_n^2(\frac{p}{j})}\,. \tag{10.48}$$

The transmission zeros are all at $p = \infty$. The poles are found from

$$1+\varepsilon^2 T_n^2(\frac{p}{j}) = 0$$

or

$$\cos(n\cos^{-1}\frac{p}{j}) = \pm\frac{1}{\varepsilon}j\,. \tag{10.49}$$

Let the poles be

$$p_k = x_k + jy_k\,.$$

Then from the solution of Eq. (10.49), we get for $k = 1, 2, \cdots, 2n$:

$$x_k = \sin\frac{(2k-1)\pi}{2n}\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right); \tag{10.50}$$

$$y_k = \cos\frac{(2k-1)\pi}{2n}\cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right). \tag{10.51}$$

Forming squares, we find:

$$\frac{x_k^2}{\left[\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)\right]^2} + \frac{y_k^2}{\left[\cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)\right]^2} = 1\,. \tag{10.52}$$

So the poles lie on an ellipse centered at the origin with major axis a and minor axis b given by

$$a = \sinh((1/n)\sinh^{-1}(1/\varepsilon))\,, \qquad b = \cosh((1/n)\sinh^{-1}(1/\varepsilon))\,. \tag{10.53}$$

We need to find the constant A. At DC, the filter, which is an LC ladder terminated at the input and output in R_1 and R_2, respectively, yields

$$|S_{21}(0)|^2 = \frac{4R_1R_2}{(R_1+R_2)^2} \tag{10.54}$$

On the other hand, we have $T_n(0) = 0$ for odd n and $T_n(0) = \pm 1$ for even n. So

$$A = \frac{4R_1R_2}{(R_1+R_2)^2} \qquad n = odd\,, \tag{10.55}$$

$$A = (1+\varepsilon^2)\frac{4R_1R_2}{(R_1+R_2)^2} \qquad n = even\,. \tag{10.56}$$

Finally, we have

$$S_{21}(p)S_{21}(-p) = \frac{A}{1+\varepsilon^2 T_n^2(\frac{p}{j})}, \tag{10.57}$$

$$S_{11}(p)S_{11}(-p) = 1 - S_{21}(p)S_{21}(-p) = (1-A)\frac{1+\eta^2 T_n^2(\frac{p}{j})}{1+\varepsilon^2 T_n^2(\frac{p}{j})}, \tag{10.58}$$

where

$$\eta^2 = \frac{\varepsilon^2}{1-A}$$

So the zeros of $S_{11}(p)S_{11}(-p)$ can be found in the same way as $S_{21}(p)S_{21}(-p)$ with ε replaced by η. The zeros also lie on an ellipse.

As before we assign all the LHP poles to $S_{11}(p)$. The zeros can be chosen from the LHP or RHP zeros. Each assignment will lead to a different input impedance $Z(s)$, obtained from

$$Z(p) = R_1 \frac{1+S_{11}(p)}{1-S_{11}(p)}$$

which is then realized as the input impedance of a lossless ladder terminated in R_2.

10.3.3 Example

Let us design a low pass filter with a Chebyshev response that meets the following specifications.

1. Passband edge is 80 KHz.
2. Passband insertion loss ripple not greater than 0.25 dB.
3. Stopband edge is 300 KHz.
4. Stopband insertion loss not less than 60 dB.
5. $R_1 = 1000\,\Omega$ and $R_2 = 4000\,\Omega$.

We define attenuation in dB as before, namely

$$\alpha(y) = -10\log|S_{21}(jy)|^2 .$$

At a zero y_k, $T_n(y_k) = 0$ and $\alpha(y_k) = -10\log A$. The insertion loss ripple in dB is

$$\alpha_{max} = \alpha(1) - \alpha(y_k) = 10\log(1+\varepsilon^2),$$

from which we get

$$\varepsilon = \sqrt{10^{0.1\alpha_{max}} - 1}\,. \tag{10.59}$$

At the stopband edge, $y = y_s$ and the insertion loss is

$$\alpha_{min} = 10\log(1+\varepsilon^2 T_n^2(y_s)) = 10\log(1+\varepsilon^2\cosh^2(n\cosh^{-1} y_s)),$$

from which we can solve for n, the order of the filter, given below:

$$n = \left\lceil \frac{\cosh^{-1}\left(\dfrac{\sqrt{10^{0.1\alpha_{min}}-1}}{\sqrt{10^{0.1\alpha_{max}}-1}}\right)}{\cosh^{-1} y_s} \right\rceil . \tag{10.60}$$

Making substitutions, we find

$$\varepsilon^2 = 0.0592537\,, \qquad n = 5\,, \qquad A = 0.64\,, \qquad \eta^2 = 0.1645937\,.$$

If we assign all the RHP zeros to $S_{11}(p)$ and noting that $R_2 > R_2$, we get

$$S_{11}(p) = \frac{-p^5 + 1.076542p^4 - 1.829471p^3 + 1.166514p^2 - 0.676069p + 0.154054}{p^5 + 1.414003p^4 + 2.249702p^3 + 1.715798p^2 + 0.989314p + 0.256757}\,,$$

and the input impedance (normalized with respect to R_1) is

$$Z(p) = \frac{2.490545p^4 + 0.420230p^3 + 2.882313p^2 + 0.313245p + 0.410811}{2p^5 + 0.337461p^4 + 4.079174p^3 + 0.549284p^2 + 1.665383p + 0.102703}\,.$$

As a check, we have $Z(0) = 4$ as expected.

Continued fraction expansion of m_1/n_2 of $Z(p)$ about $p = \infty$ yields an LC ladder shown in Fig. 10.7 with the following element values after de-normalization with respect to ω_p and R_1.

Element	Element value
C_1	1.597591 nF
L_2	2.807926 mH
C_3	3.519709 nF
L_4	3.259893 mH
C_5	2.947657 nF

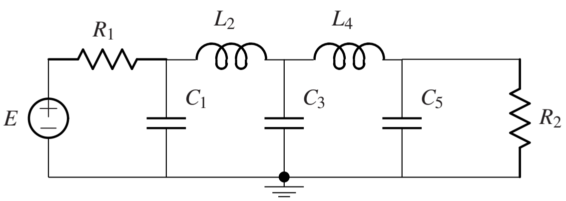


Fig. 10.7 A fifth order Chebyshev low-pass filter with $R_1 = 1000\,\Omega$ and $R_2 = 4000\,\Omega$

The frequency response as computed by node analysis is shown in Fig. (10.8). It is seen that the insertion loss meets the filter specifications. Comparing this design with that of the maximally flat filter of the last section, we see the Chebyshev filter has a larger stop-band insertion loss for the same passband ripple and the same number of elements.

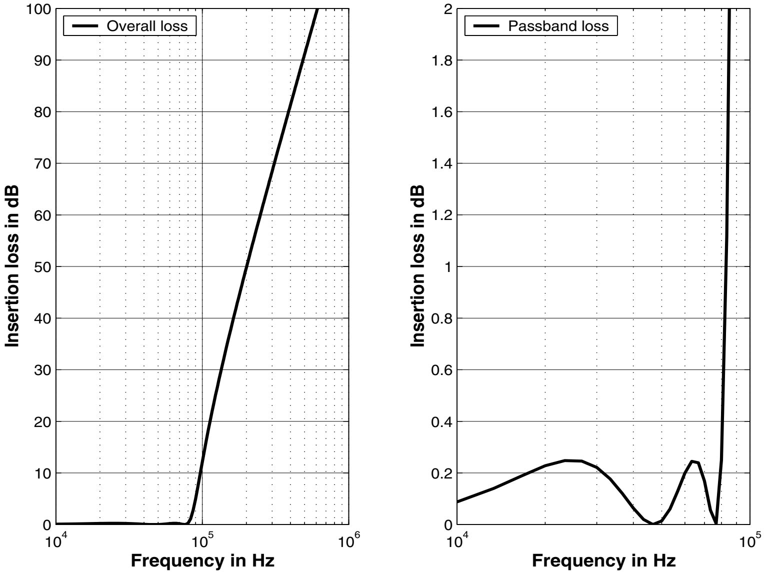


Fig. 10.8 Insertion loss of a fifth order Chebyshev low-pass filter

10.4 Elliptic filters

In both the maximally flat and Chebyshev filters, the transmission function $|S_{21}(j\omega)|^2$ has all its zeros at infinity. The function goes to zeros from the passband to the stop band smoothly as some power of ω and the transition is gradual.

It would be desirable if the transition could be made more steep. Intuitively, if we could place the transmission zeros at finite frequencies instead of infinity, we would get a filter characteristic more like that of an ideal low pass filter. In fact, what we desire is a filter with the following characteristics:

1. In the passband, the amplitude should approximate a constant in the manner of equi-ripple, namely, the maximum deviation from a constant is minimized.
2. In the stop-band, the amplitude should approximate zero in the manner of equi-ripple, namely, the maximum deviation from zero is minimized.

The elliptic filter, also called the Cauer filter, has these characteristics. Figure 10.9 shows the frequency response of the transmission function of a 5th order filter.

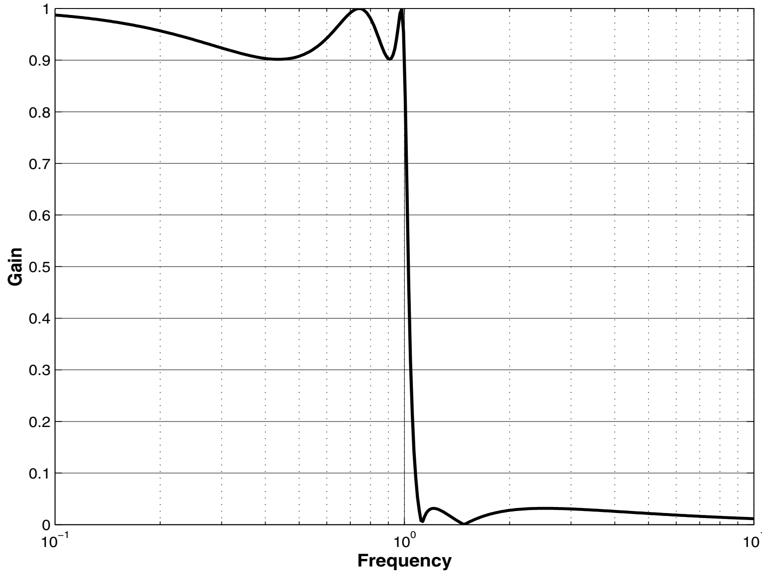


Fig. 10.9 Elliptic filter characteristic of order 5 with equi-ripples in both the passband and stopband

10.4.1 *Equi-ripple rational function*

To satisfy the requirements stated above, we propose that the transmission power gain have the following form:

$$|S_{21}(j\omega)|^2 = \frac{A}{1+\varepsilon^2 R_n^2(\omega,L)}\,, \tag{10.61}$$

where ε is a small constant and L a large constant, both being design parameters and $R_n(\omega,L)$ is a rational function of ω of order n:

$$R_n(\omega,L) = C_1\omega \prod_k^{(n-1)/2} \frac{\omega^2-\omega_k^2}{\omega^2-\Omega_k^2}, \quad \text{for } n = \text{odd}\,, \tag{10.62}$$

$$R_n(\omega,L) = C_2 \prod_k^{n/2} \frac{\omega^2-\omega_k^2}{\omega^2-\Omega_k^2}, \quad \text{for } n = \text{even}\,. \tag{10.63}$$

We choose the poles Ω_k and zeros ω_k such that

$$\max|S_{21}(j\omega)|^2 = A\,, \quad \text{for } |\omega| \le \omega_p\,; \tag{10.64}$$

$$\min|S_{21}(j\omega)|^2 = \frac{A}{1+\varepsilon^2}\,, \quad \text{for } |\omega| \le \omega_p\,; \tag{10.65}$$

$$\max|S_{21}(j\omega)|^2 = \frac{A}{1+\varepsilon^2 L^2}\,, \quad \text{for } |\omega| \ge \omega_s\,; \tag{10.66}$$

$$\min|S_{21}(j\omega)|^2 = 0\,, \quad \text{for } |\omega| \ge \omega_s\,; \tag{10.67}$$

where ω_p is the passband edge and ω_s is the stop-band edge. It follows from the above that the maxima in the passband are attained at the zeros ω_k, and the minima in the stop-band at the poles Ω_k. So it is sufficient to choose all the zeros to be in the passband and all the poles in the stop-band, namely, $\omega_k \le \omega_p$ and $\Omega_k \ge \omega_s$.

Let the frequencies at which Eq. (10.65) is satisfied be ω_k'. Then we require

$$R_n(\omega_k', L) = \pm 1\,. \tag{10.68}$$

Let the frequencies at which Eq. (10.66) is satisfied be Ω_k'. Then

$$R_n(\Omega_k', L) = \pm L\,. \tag{10.69}$$

As before, it is convenient to normalize the frequency variable with respect to the passband edge ω_p. Let

$$p = x + jy = \frac{s}{\omega_p} = \frac{\sigma}{\omega_p} + j\frac{\omega}{\omega_p} \tag{10.70}$$

so that the passband edge is $y = 1$ and the stop-band edge is $y_s = \omega_s/\omega_p$. Denote

$$y_k = \omega_k/\omega_p\,, \qquad y_k' = \omega_k'/\omega_p\,, \tag{10.71}$$

$$Y_k = \Omega_k/\omega_p\,, \qquad Y_k' = \Omega_k'/\omega_p\,. \tag{10.72}$$

The requirements on $|S_{21}(j\omega)|^2$ can be stated in terms of $R_n(y,L)$, as follows.

$$R_n(y,L) = C_1 y \prod_{k=1}^{(n-1)/2} \frac{y^2 - y_k^2}{y^2 - Y_k^2}, \quad \text{for } n = \text{odd}\,; \tag{10.73}$$

$$R_n(y,L) = C_2 \prod_{k=1}^{n/2} \frac{y^2 - y_k^2}{y^2 - Y_k^2}, \quad \text{for } n = \text{even}\,; \tag{10.74}$$

$$|y_k| \le 1\,; \tag{10.75}$$

$$|Y_k| \ge y_s\,; \tag{10.76}$$

$$|R_n(y,L)| \le 1 \quad \text{for } |y| \le 1\,; \tag{10.77}$$

$$|R_n(y,L)| \ge L \quad \text{for } |y| \ge y_s\,. \tag{10.78}$$

The equi-ripple requirements in the passband and stop-band, expressed in Eqs. (10.77) and (10.78), can be simultaneously satisfied if we impose an additional condition on $R_n(y,L)$, namely

$$R_n(y,L) = \frac{L}{R_n(\frac{y_s}{y},L)}\,, \tag{10.79}$$

which means that if y_k is a zero of $R_n(y,L)$, then

$$Y_k = \frac{y_s}{y_k} \tag{10.80}$$

is automatically a pole of $R_n(y,L)$. Figure 10.10 shows a plot of $R_5(y,L)$. It is seen that $R_5(y,L)$ is equi-rippled in the passband ($y = \pm 1$) and its maximal deviation from zero are all equal in the stop-band ($|y| \geq y_s$). There are five zeros in the passband and five poles in the stop-band (one at $y = \infty$). We will show later that in fact the

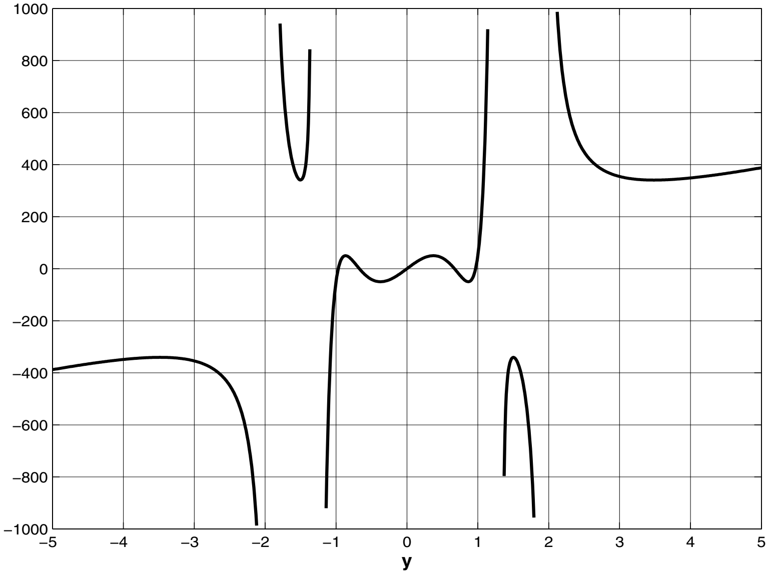


Fig. 10.10 Rational function $R_5(y,L)$. Note the equi-ripples for $|y| \leq 1$ and for $|y| \geq y_s$

equi-ripple rational function $R_n(y,L)$ is given by:

$$R_n(y,L) = C_1 y \prod_{\nu=1}^{(n-1)/2} \frac{y^2 - sn^2\left(\frac{2\nu K}{n}, \frac{1}{y_s}\right)}{y^2 - \dfrac{y_s^2}{sn^2\left(\frac{2\nu K}{n}, \frac{1}{y_s}\right)}}, \quad \text{n=odd}; \tag{10.81}$$

$$R_n(y,L) = C_2 \prod_{\nu=1}^{n/2} \frac{y^2 - sn^2\left(\frac{(2\nu-1)K}{n}, \frac{1}{y_s}\right)}{y^2 - \dfrac{y_s^2}{sn^2\left(\frac{(2\nu-1)K}{n}, \frac{1}{y_s}\right)}}, \quad \text{n=even}; \tag{10.82}$$

where

$$sn(u,k) = z$$

is the Jacobi elliptic sine function of modulus k. Its inverse

$$sn^{-1}(z,k) = u$$

is defined as

$$u(z,k) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = sn^{-1}(z,k), \quad (k^2 < 1), \tag{10.83}$$

and is called the incomplete elliptic integral of the first kind. The complete elliptic integral of the first kind is

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \quad (k^2 < 1). \tag{10.84}$$

We should compare these functions with something more familiar. For example

$$u(z) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \sin^{-1} z, \qquad \sin u = z. \tag{10.85}$$

$$u(1) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \frac{\pi}{2}, \qquad \sin\frac{\pi}{2} = 1. \tag{10.86}$$

Values of the elliptic sine function and complete elliptic integral are tabulated in standard mathematical tables[1], or available as functions in programming languages.[1]

We also need the complementary complete elliptic integral of the first kind:

$$K'(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-(k')^2t^2}} = K(k'), \tag{10.87}$$

where

$$k' = \sqrt{1-k^2}$$

[1] In MATLAB®, $y = ellipj(u,m)$ yields $sn(u,k)$ and $y = ellipke(m)$ yields $K(k)$, where $m = k^2$.

is called the complementary modulus. We will study the properties of elliptic functions and elliptic integrals later when we derive the equi-ripple rational function $R_n(y,L)$.

10.4.2 Design formulas

We define attenuation in dB as before, namely

$$\alpha(y) = -10\log|S_{21}(jy)|^2$$

and the insertion loss ripple is

$$\alpha_{max} = \alpha(1) - \alpha(y_k) = 10\log(1+\varepsilon^2)\,, \tag{10.88}$$

where y_k is a zero of $R_n(y,L)$ in the passband. It follows that

$$\varepsilon^2 = 10^{0.1\alpha_{max}} - 1\,. \tag{10.89}$$

At the stop-band edge, $y = y_s$. The minimum insertion loss in the stop band is

$$\alpha_{min} = \alpha(y_s) - \alpha(y_k) = 10\log(1+\varepsilon^2 L^2)\,. \tag{10.90}$$

Combining Eqs. (10.89) and (10.90), we get an expression for the parameter L in terms of the design parameters α_{max} and α_{min}:

$$L = \left[\frac{10^{0.1\alpha_{min}} - 1}{10^{0.1\alpha_{max}} - 1}\right]^{1/2}\,. \tag{10.91}$$

It will be shown later that the order of the filter n is given by

$$n = \left\lceil \frac{K(k_s)K'(k_L)}{K'(k_s)K(k_L)} \right\rceil\,, \tag{10.92}$$

where the moduli are:

$$k_s = \frac{1}{y_s}\,, \qquad k_L = \frac{1}{L}\,. \tag{10.93}$$

Given the passband ripple α_{max}, the minimum stop-band attenuation α_{min}, and the ratio of the passband edge to the stop-band edge, we can compute the order of the filter required, the locations of the zeros and poles of the equi-ripple rational function $R_n(y,L)$, and finally the transmission function $|S_{21}(j\omega)|^2$, which will have transmission zeros at finite frequencies. We then follow the procedure of the last chapter to synthesize a two-port terminated in resistors at both ends to realize a low-pass filter with equi-ripple characteristics in both the passband and stop-band.

10.4.3 Example

Let us find a low-pass filter with the following specifications.

1. Passband ripple: $\alpha_{max} = 0.1$ dB;
2. Minimum attenuation in the stop-band: $\alpha_{min} = 30$ dB;
3. Passband edge: $\omega_p = 1$.
4. Stop-band edge: $\omega_s = 1.3$, and we have $y_s = 1.3$.
5. The filter is to be terminated in $1\,\Omega$ at the input and output ports.

The transition from passband to stop-band is relatively steep. We first compute parameter L from Eq. (10.91):

$$L = 207.095233\,.$$

Next the modulii are computed:

$$k_s = \frac{1}{y_s} = 0.769230\,, \qquad k_s' = \sqrt{1-k_s^2} = 0.638971\,,$$
$$k_L = \frac{1}{L} = 0.00482869\,, \qquad k_L' = \sqrt{1-k_L^2} = 0.999988\,,$$

followed by the complete elliptic integrals:

$$K(k_s) = 1.940716\,, \qquad K'(k_s) = K(k_s') = 1.783308\,,$$
$$K(k_L) = 1.570805\,, \qquad K'(k_L) = K(k_L') = 6.719506\,.$$

Finally, the order of the filter is

$$n = \left\lceil \frac{K(k_s)K'(k_L)}{K'(k_s)K(k_L)} \right\rceil = 5\,. \tag{10.94}$$

We now have all the information to construct the equi-ripple rational function $R_5(y,L)$:

$$R_5(y,L) = C_1 y \frac{(y^2-y_1^2)(y^2-y_2^2)}{(y^2-Y_1^2)(y^2-Y_2^2)}\,, \tag{10.95}$$

where the zeros y_k and poles Y_k are:

$$y_1 = sn\frac{2K(k_s)}{5} = 0.671178\,, \qquad y_2 = sn\frac{4K(k_s)}{5} = 0.968498\,, \tag{10.96}$$
$$Y_1 = \frac{y_s}{y_1} = 1.936892\,, \qquad Y_2 = \frac{y_s}{y_2} = 1.342284\,. \tag{10.97}$$

To determine the constant C_1 in Eq. (10.95), we make use of the fact that at the passband edge ($y = 1$), $R_n(y,L) = 1$. So

$$R_5(1,L) = 1 = C_1 \frac{(1-y_1^2)(1-y_2^2)}{(1-Y_1^2)(1-Y_2^2)}\,, \tag{10.98}$$

and

$$C_1 = \frac{(1-Y_1^2)(1-Y_2^2)}{(1-y_1^2)(1-y_2^2)} = 64.737240\,. \tag{10.99}$$

We will also need the following constant:

$$\varepsilon^2 C_1^2 = 97.618840\,. \tag{10.100}$$

Lastly, the constant A is determined from the transmission power gain at $\omega = 0$, as follows.

$$|S_{21}(jy)|^2 = 4\frac{R_1}{R_2}\left|\frac{V_{out}}{E}\right|^2 = \frac{A}{1+\varepsilon^2 R_n^2(y,L)}\,. \tag{10.101}$$

Evaluating at $y = 0$, we get

$$A = \begin{cases} 4\dfrac{R_1}{R_2}\left(\dfrac{R_2}{R_1+R_2}\right)^2, & \text{n=odd;} \\ (1+\varepsilon^2)\left(4\dfrac{R_1}{R_2}\right)\left(\dfrac{R_2}{R_1+R_2}\right)^2, & \text{n=even}\,, \end{cases} \tag{10.102}$$

where R_1 and R_2 are the terminating resistances at port-1 and port-2, respectively (not to be confused with the rational function $R_n(y)$). In our case, $A = 1$.

The transmission function is then, from Eqs. (10.95) and (10.101):

$$|S_{21}(jy)|^2 = \frac{(y^2-Y_1^2)^2(y^2-Y_2^2)^2}{(y^2-Y_1^2)^2(y^2-Y_2^2)^2+\varepsilon^2 C_1^2 y^2(y^2-y_1^2)^2(y^2-y_2^2)^2}\,. \tag{10.103}$$

It is seen that the transmission zeros are at finite frequencies $Y_1 = 1.936892$ and $Y_2 = 1.342284$, and they are the poles of $R_5(y,L)$.

Following the procedure outlined in Sect. 9.4.2 for the synthesis of transfer functions with transmission zeros at finite frequencies, and choosing all the zeros and poles of $S_{11}(p)$ to be in the left-half s-plane, we obtain the reflection coefficient and input impedance function given by, respectively:

$$S_{11}(p) = \frac{-p^5 - 1.388470p^3 - 0.422546p}{p^5+1.699967p^3+2.828292p^3+2.578953p^2+1.714101p+0.684121} \tag{10.104}$$

$$Z(p) = \frac{1.699967p^4+1.439821p^3+2.578953p^2+1.291555p+0.684121}{2p^5+1.699967p^4+4.216761p^3+2.578953p^2+2.136647p+0.684121} \tag{10.105}$$

We remove the transmission zero sections in the order: (Y_2, Y_1), in accordance with Fujisawa's guidelines (Sect. 9.4.3). The realized filter is shown in Fig. 10.11, in which

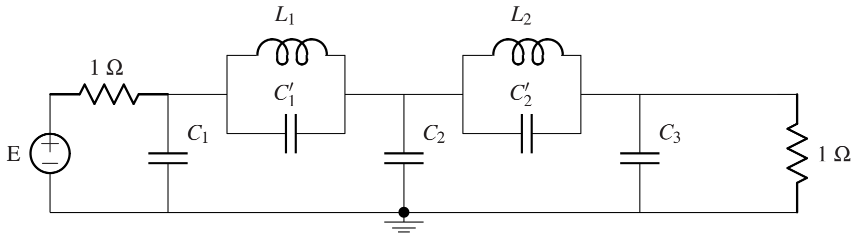


Fig. 10.11 A fifth order elliptic low-pass filter

$$
\begin{aligned}
C_1 &= 0.662996\,\mathrm{F}, & C_2 &= 1.491613\,\mathrm{F}, & C_3 &= 0.968584\,\mathrm{F}, \\
C_1' &= 0.739479\,\mathrm{F}, & C_2' &= 0.234367\,\mathrm{F}, & & \\
L_1 &= 0.750560\,\mathrm{H}, & L_2 &= 1.137346\,\mathrm{H}. & &
\end{aligned}
$$

Figure 10.12 shows the insertion loss of the filter. The equi-ripple characteristics of both the passband and stop-band are evident. The minimum insertion loss in the stop-band is about 34 dB, which is greater than the desired value. This comes about because n, being an integer, is greater than what is required to provide an insertion loss of 30 dB. See Eq. (10.94).

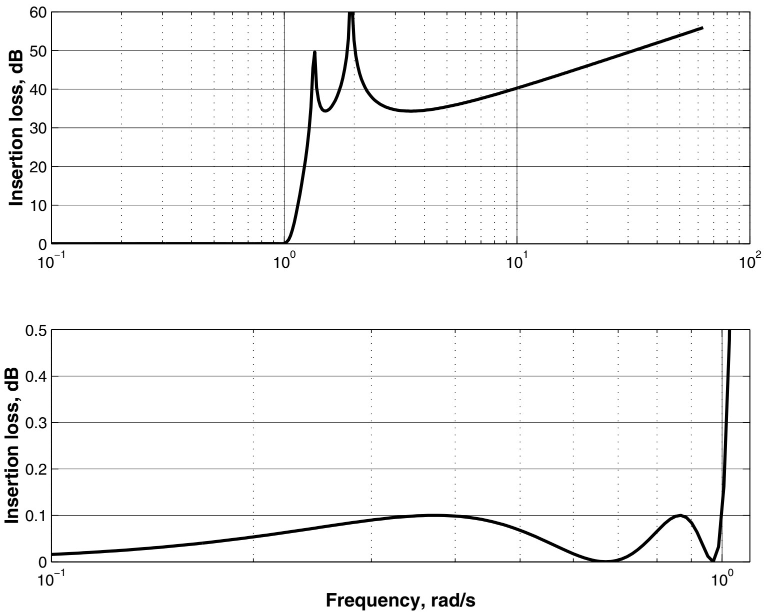


Fig. 10.12 Frequency response of 5th order elliptic low-pass filter overall (upper) and in the passband (lower)

10.4.4 Derivation of $R_n(y, L)$

It remains to derive the equi-ripple rational function $R_n(y,L)$. As detailed in [18], we first obtain a differential equation for $R_n(y,L)$, solve it in terms of elliptic sine functions, and then deduce the locations of zeros of $R_n(y,L)$, those of poles being automatically determined from Eq. (10.80).

We note the following about $R_n(y,L)$:

1. $R_n(y,L)$ is an odd function of y for n =odd. It is even for n =even.
2. $R_n(y,L)$ has all its n zeros in the passband: $-1 \leq y \leq 1$.
3. $R_n(y,L)$ has all its poles in the stop-band: $|y| > y_s$.
4. $R_n(y,L)$ oscillates between -1 and $+1$ in the passband.
5. At the passband edge, $y = 1$ and $R_n(1,L) = 1$. At $y = -1$, $R_n(-1,L) = (-1)^n$.
6. At the stop-band edge, $y = y_s$ and $R_n(y_s,L) = L$. $R(-y_s,L) = (-L)^n$.
7. $R_n(y,L)$ oscillates between L and $-L$ in the stop-band.
8. $\mathrm{d}R_n/\mathrm{d}y$ has $n-1$ zeros in the passband where $R_n = \pm 1$.
9. $\mathrm{d}R_n/\mathrm{d}y$ has $n-1$ zeros in the stop-band where $R_n = \pm L$.
10. $(\mathrm{d}R_n/\mathrm{d}y)^2$ has $4(n-1)$ zeros. (A second order zero is counted twice.) The zeros are of second order and occur at $R_n = \pm 1$ and $R_n = \pm L$.
11. For n =odd, $R_n + 1$ has $(n-1)/2$ double zeros at the minima of R_n in the passband and one simple zero at $y = -1$. For n =even, $R_n + 1$ has $n/2$ double zeros in the passband. In either case, $R_n + 1$ has n zeros in the passband.
12. $R_n - 1$ has $(n-1)/2$ double zeros at the maxima of R_n in the passband and one simple zero at $y = 1$, for n =odd. For n =even, $R_n - 1$ has $(n-2)/2$ double zeros and one simple zeros each at $y = \pm 1$. All together there are n zeros.
13. Similarly, $R_n + L$ and $R_n - L$ each have n zeros in the stop-band.
14. $(R_n^2 - 1)(R_n^2 - L^2)$ has $4n$ zeros and they occur at $R_n = \pm 1$ and $R_n = \pm L$. The zeros at $y = \pm 1$ and $y = \pm y_s$ are of first order.

It follows that

$$\left(\frac{\mathrm{d}R_n}{\mathrm{d}y}\right)^2 \quad \text{and} \quad \frac{(R_n^2 - 1)(R_n^2 - L^2)}{(y^2 - 1)(y^2 - y_s^2)}$$

have the same zeros and (poles). Thus we can write

$$\frac{\mathrm{d}R_n}{\mathrm{d}y} = \pm M \sqrt{\frac{(1 - R_n^2)(L^2 - R_n^2)}{(1 - y^2)(y_s^2 - y^2)}}, \tag{10.106}$$

where M is a constant. Separation of variables leads to

$$\frac{C\,\mathrm{d}R_n}{\sqrt{1 - R_n^2}\sqrt{L^2 - R_n^2}} = \frac{\mathrm{d}y}{\sqrt{1 - y^2}\sqrt{y_s^2 - y^2}} = \mathrm{d}u, \tag{10.107}$$

where C is a constant and u is an intermediate variable that connects R_n to y. We recall the incomplete elliptic integral of the first kind is defined as

$$u(z,k) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,; \tag{10.108}$$

$$z = sn(u,k)\,. \tag{10.109}$$

It follows that

$$y = sn(y_s u + b_1, k_s)\,, \tag{10.110}$$

where $k_s = 1/y_s$ is the modulus associated with y and b_1 is a constant of integration. Similarly

$$R_n = sn(\frac{L}{C}u + b_2, k_L)\,, \tag{10.111}$$

where $k_L = 1/L$ is the modulus associated with R_n, and b_2 is a constant of integration. To determine b_1 and b_2, we invoke the boundary conditions. At $y = 0$, we want $u = 0$, and we have $b_1 = 0$. Since

$$R_n(y,L)|_{y=0} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{n/2}, & \text{if } n \text{ is even.} \end{cases} \tag{10.112}$$

and since the complete elliptic integral $K(k)$ is

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,,$$

we have

$$b_2 = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{n/2}K(k_L), & \text{if } n \text{ is even.} \end{cases} \tag{10.113}$$

In summary, the solution to Eq. (10.107) is

$$R_n(y,L) = \begin{cases} sn\left(\frac{L}{C}u, k_L\right) & \text{if } n \text{ is odd;} \\ sn\left(\frac{L}{C}u + (-1)^{n/2}K(k_L), k_L\right) & \text{if } n \text{ is even.} \end{cases} \tag{10.114}$$

where

$$y = sn(y_s u, k_s)\,. \tag{10.115}$$

It is seen that $R_n(y,L)$ is not explicitly given in terms of y, but rather indirectly through the intermediate variable u. Given a value of y, we first find the corresponding value of u from Eq. (10.115), and then use it in Eq. (10.114) to obtain the value of $R_n(y,L)$. This is most inconvenient for our purpose and we must develop a more direct expression, preferably in terms of poles and zeros.

10.4.5 Elliptic functions

We need a few preliminaries. By a change of variables in Eq. (10.83) ($t = \sin x$ and $z = \sin\phi$), we get an alternative definition of the elliptic sine function:

$$u(\phi,k) = \int_0^{\phi} \frac{\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}}\,, \tag{10.116}$$

and the elliptic sine function is

$$sn(u,k) = \sin\phi\,. \tag{10.117}$$

We can define additional functions: the elliptic cosine function

$$cn(u,k) = \cos\phi\,. \tag{10.118}$$

So we have

$$sn^2(u,k) + cn^2(u,k) = 1\,; \tag{10.119}$$

and the difference function

$$dn(u,k) = \frac{\mathrm{d}\phi}{\mathrm{d}u} = \sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2 sn^2(u,k)}\,. \tag{10.120}$$

The elliptic sine and cosine functions are plotted in Fig. 10.13.

The following are useful properties:[18]

$$sn(u,k) = -sn(-u,k)\,, \qquad cn(u,k) = cn(-u,k)\,.$$

$$\begin{aligned}
&sn(0,k)=0\,, \quad sn(K,k)=1\,, \quad sn(2K,k)=0\,, \quad sn(3K,k)=-1\,, \quad sn(4K,k)=0\,.\\
&cn(0,k)=1\,, \quad cn(K,k)=0\,, \quad cn(2K,k)=-1\,, \quad cn(3K,k)=0\,, \quad cn(4K,k)=1\,.
\end{aligned}$$

$$\begin{aligned}
sn(u+K,k) &= sn(K-u,k)\,, & cn(u+K,k) &= -cn(K-u,k)\,.\\
sn(u+2K,k) &= -sn(u,k)\,, & cn(u+2K,k) &= -cn(u,k)\,.\\
sn(u+4K,k) &= sn(u,k)\,, & cn(u+4K,k) &= cn(u,k)\,.
\end{aligned}$$

So $sn(u,k)$ is an odd function of u and $cn(u,k)$ is even. For u real, $sn(u,k)$ is periodic with a period of $4K(k)$. We also have the following identities:

$$sn(u+v) = \frac{sn\,u\cdot cn\,v\cdot dn\,v + sn\,v\cdot cn\,u\cdot dn\,u}{1-k^2 sn^2 u\cdot sn^2 v}\,. \tag{10.121}$$

$$cn(u+v) = \frac{cn\,u\cdot cn\,v - sn\,u\cdot sn\,v\cdot dn\,u\cdot dn\,v}{1-k^2 sn^2 u\cdot sn^2 v}\,. \tag{10.122}$$

$$sn(u+v)\cdot sn(u-v) = \frac{sn^2 u - sn^2 v}{1-k^2 sn^2 u\cdot sn^2 v}\,. \tag{10.123}$$

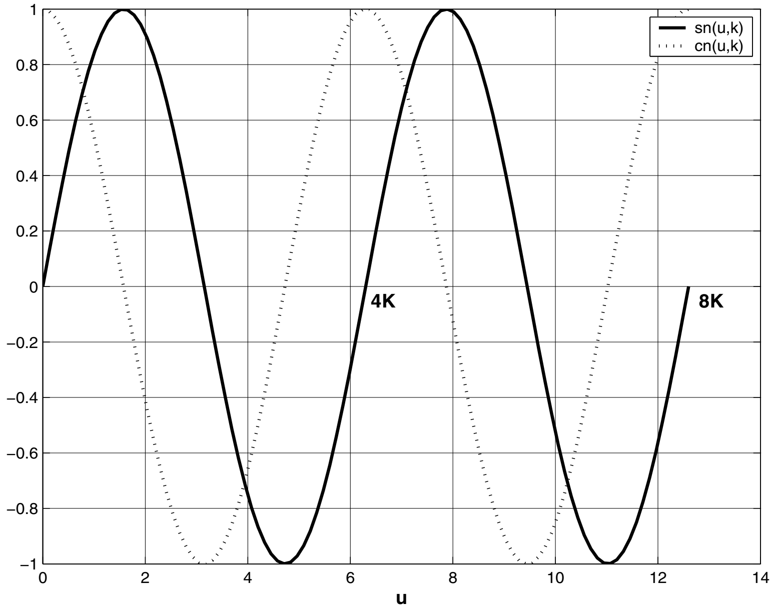


Fig. 10.13 Elliptic sine and cosine functions with period $4K(k)$ and range ± 1

For u imaginary, we have the following properties:

$$sn(iu,k) = i\frac{sn(u,k')}{cn(u,k')}. \qquad sn(iu,k) = -sn(-iu,k).$$

$$sn(iK',k) = \infty. \qquad sn(i(u+K'),k) = -sn(i(K'-u),k).$$

$$sn(i(u+2K'),k) = sn(iu,k). \qquad sn(iu,1) = i\tan u.$$

So $sn(u,k)$ is a doubly-periodic function with a real period of $4K$ and an imaginary period of $2K'$. Figure 10.14 shows a plot of $sn(iu,k)/i$.

For complex argument $u = v + iw$, where $v = \Re u$ and $w = \Im u$, we have the following identity:

$$sn(v+iw,k) = \frac{sn(v,k)\,dn(w,k') + i\,cn(v,k)\,dn(v,k)\,sn(w,k')\,cn(w,k')}{cn^2(w,k') + k^2 sn^2(v,k)\,sn^2(w,k')}. \quad (10.124)$$

The properties of $sn(u,k)$ are best summarized in a "periodic rectangle" in the complex u-plane. The rectangle has length $4K$ along the v-axis and height $2K'$ along the w-axis. Fig. 10.15 shows the rectangle and the values of $sn(u,k)$ at critical points. The rectangle is repeated in the entire complex plane every $4K$ along v and every $2K'$ along w.

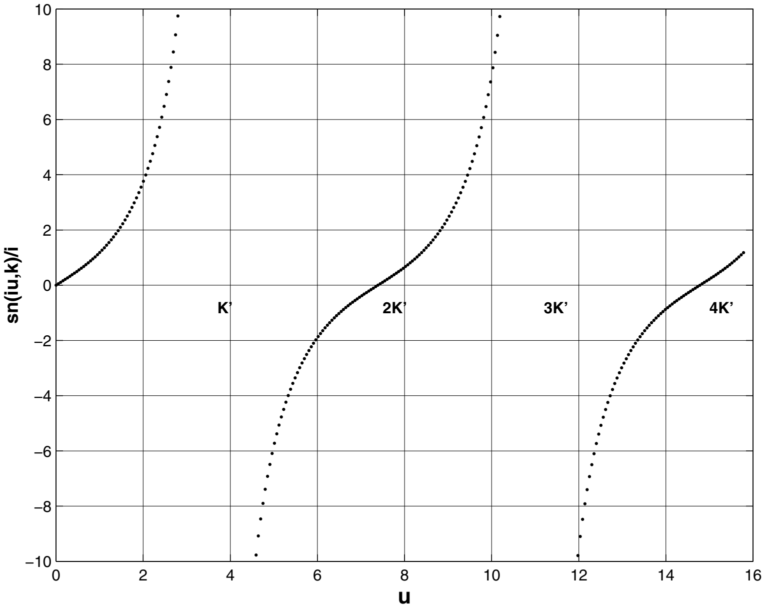


Fig. 10.14 Elliptic sine of imaginary argument with period $2K^{'}$

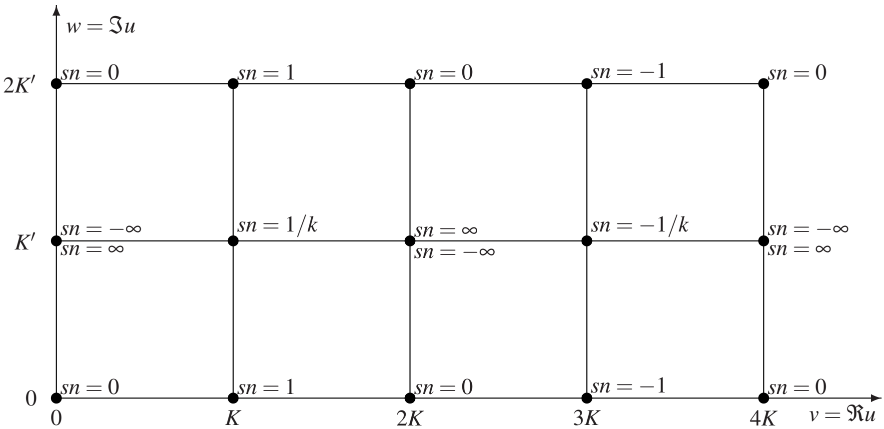


Fig. 10.15 Periodic rectangle of elliptic sine function $sn(u,k)$

10.4.6 Periodic rectangles of $R_n(y,L)$ *and* $sn(y_s u, k_s)$

Returning to the derivation of $R(y,L)$, we recall the solution to Eq. (10.107) is

$$R_n(y,L) = \begin{cases} sn\left(\frac{L}{C}u, k_L\right) & \text{if } n \text{ is odd;} \\ sn\left(\frac{L}{C}u + (-1)^{n/2}K(k_L), k_L\right) & \text{if } n \text{ is even}, \end{cases} \tag{10.125}$$

where

$$y = sn(y_s u, k_s)\,. \tag{10.126}$$

Let us display the periodic rectangle of $R_n(y,L)$ for n = odd. (It is similar for n = even except for a shift.) This is shown in Fig. 10.16. The periodic rectangle

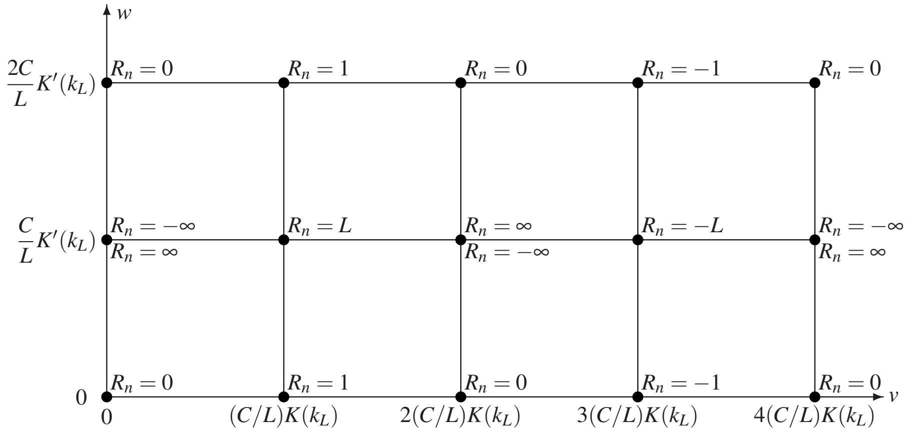


Fig. 10.16 Periodic rectangle of $R_n(y,L) = sn(L/C\,u, k_L)$ for n = odd

of $y = sn(y_s u, k_s)$ is shown in Fig. 10.17.

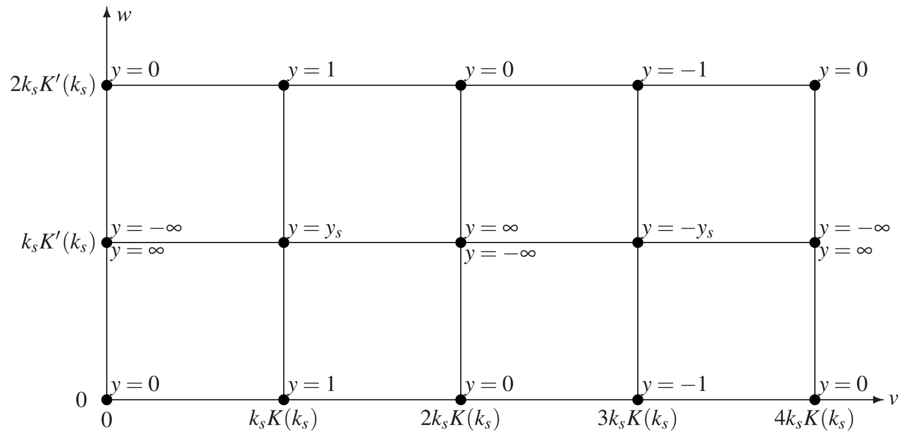


Fig. 10.17 Periodic rectangle of $y = sn(y_s u, k_s)$, $k_s = 1/y_s$

Recall that u is an intermediate variable common to both y and R_n. In the passband, as y varies from 0 to the passband edge $y = 1$, R_n varies, for n = 5, from 0 to 1, 1 to 0, 0 to -1, -1 to 0, and then 0 to 1, going through such variations 5 times. Correspondingly, with reference to Fig. 10.17, u ranges from 0 to $k_sK(k_s)$, and with reference to Fig. 10.16, u ranges from 0 to $5(C/L)K(k_L)$. Since the range of u must be the same in both, we obtain an important relation:

$$k_s K(k_s) = \frac{nC}{L} K(k_L). \tag{10.127}$$

In the transition band, y varies from the passband edge $y = 1$ to the stop-band edge $y = y_s$, or with reference to Fig. 10.17, $w = \Im u$ ranges from 0 to $k_s K'(k_s)$. Correspondingly, R_n varies from 1 to L in the transition band and with reference to Fig. 10.16, $w = \Im u$ ranges from 0 to $(C/L)K'(k_L)$. It follows that

$$k_s K'(k_s) = \frac{C}{L} K'(k_L). \tag{10.128}$$

Combining the two equations, we get one of the design equations of elliptic filters:

$$n = \left\lceil \frac{K(k_s)K'(k_L)}{K'(k_s)K(k_L)} \right\rceil \tag{10.129}$$

as before.

10.4.7 Recalculation of L

Since the order of the filter n is the smallest integer greater than the fraction on the right-hand side of Eq. (10.129), if we are to satisfy the relation

$$K(k_s)K'(k_L) = nK'(k_s)K(k_L) \tag{10.130}$$

for a given n and $k_s = 1/y_s$, the stop-band parameter L will have to change from its original value determined from α_{min} and α_{max} in Eq. (10.91). In fact, L can be expressed explicitly in terms of n and k_s as the following will show.

Theorem 10.1. *Let $y = sn(y_s u, k_s)$. Then $R_n(y,L)$ can be expressed as*

$$n = odd: \quad R_n(y,L) = m \prod_{p=0}^{n-1} sn\left(y_s u + \frac{2p}{n} K(k_s), k_s\right), \tag{10.131}$$

$$n = even: \quad R_n(y,L) = m \prod_{p=0}^{n-1} sn\left(y_s u + \frac{2p+1}{n} K(k_s), k_s\right), \tag{10.132}$$

(10.133)

where m is a constant.

Proof. Let us do the case of n = odd only. We begin with $R_n(y,L) = sn(\frac{L}{C}u, k_L)$. Its zeros are located at

$$\begin{aligned} u_z &= 2p\frac{C}{L}K(k_L) \\ &= \frac{2p}{n}k_sK(k_s), \quad p = 0, 1, 2, \ldots \end{aligned} \tag{10.134}$$

because of (10.127). In Eq. (10.131), the zeros of the right-hand side expression are:

$$u_z = -\frac{2p}{n}k_sK(k_s), \quad p = 0, 1, 2, \ldots \tag{10.135}$$

which agree with those of Eq. (10.134) since $sn(u,k)$ is an odd function of u. Let us look at the poles. From the periodic rectangle of $R_n(y,L) = sn(\frac{L}{C}u, k_L)$, we find the poles to be:

$$\begin{aligned} u_p &= 2p\frac{C}{L}K(k_L) + i(2p+1)\frac{C}{L}K'(k_L), \quad p = 0, 1, 2, \ldots \\ &= \frac{2p}{n}k_sK(k_s) + i(2p+1)k_sK'(k_s) \end{aligned} \tag{10.136}$$

because of (10.127) and (10.128). From the periodic rectangle of $sn(y_su, k_s)$, we find the poles of the expression of the right-hand side of Eq. (10.134) to be:

$$u_p = -\frac{2p}{n}k_sK(k_s) + i(2p+1)k_sK'(k_s), \quad p = 0, 1, 2, \ldots \tag{10.137}$$

which are the same as those of $R_n(y,L) = sn(\frac{L}{C}u, k_L)$. So the right-hand side expression of Eq. (10.131) has the same poles and zeros as $sn(\frac{L}{C}u, k_L) = R_n(y,L)$ and hence they are equal except for a multiplicative constant m. □

Next, from the periodic rectangle of R_n (n = odd), we find

$$R_n(y,L)\big|_{u=\frac{C}{L}K(k_L)} = 1\,. \tag{10.138}$$

$$R_n(y,L)\big|_{u=\frac{C}{L}K(k_L)+i\frac{C}{L}K'(k_L)} = L\,. \tag{10.139}$$

Using (10.127) and (10.128) again, we get

$$1 = R_n(y,L)\big|_{u=k_sK(k_s)/n}\,. \tag{10.140}$$

$$L = R_n(y,L)\big|_{u=k_sK(k_s)/n+ik_sK'(k_s)}\,. \tag{10.141}$$

Applying (10.140) to (10.131), we obtain

$$m^{-1} = \prod_{p=0}^{n-1} sn\left(\frac{2p+1}{n}K(k_s), k_s\right). \tag{10.142}$$

Applying (10.141), we get

$$\frac{1}{L} = \frac{\prod_{p=0}^{n-1} sn\left(\frac{2p+1}{n}K(k_s),k_s\right)}{\prod_{p=0}^{n-1} sn\left(\frac{2p+1}{n}K(k_s)+iK'(k_s),k_s\right)}. \tag{10.143}$$

But we have the identity:

$$sn(u+iK'(k),k) = \frac{1}{k\,sn(u,k)}. \tag{10.144}$$

Using it in Eq. (10.143) and noting $sn(2K-u,k) = sn(u,k)$, we obtain an expression for L in terms of n and k_s:

$$\begin{aligned}\frac{1}{L} &= y_s \prod_{p=0}^{n-1} sn^2\left(\frac{2p+1}{n}K(k_s),k_s\right)\\ &= y_s \prod_{p=0}^{\lceil n/2\rceil-1} sn^4\left(\frac{2p+1}{n}K(k_s),k_s\right).\end{aligned} \tag{10.145}$$

10.4.8 Rational expression of $R_n(y,L)$

Returning to the task of expressing $R_n(y,L)$ as a rational function of y, we refer to Eq. (10.131) and expand the product term by term. Again making use of the identity $sn(2K-u,k) = sn(u,k)$, we have

n = odd:

$$R_n(y,L) = m\,sn(y_s u,k_s)\prod_{p=1}^{(n-1)/2} sn\left(\frac{2p}{n}K(k_s)+y_s u\right) sn\left(\frac{2p}{n}K(k_s)-y_s u\right). \tag{10.146}$$

n = even:

$$R_n(y,L) = m\prod_{p=1}^{n/2} sn\left(\frac{2p-1}{n}K(k_s)+y_s u\right) sn\left(\frac{2p-1}{n}K(k_s)-y_s u\right). \tag{10.147}$$

Using identity (10.123) and noting $y = sn(y_s u,k_s)$, we obtain the rational expression for $R_n(y,L)$ in terms of y given below:

$$\text{n = odd:}\quad R_n(y,L) = C_1 y \prod_{p=1}^{(n-1)/2} \frac{y^2 - sn^2\left(\frac{2p}{n}K(k_s),k_s\right)}{y^2 - \dfrac{y_s^2}{sn^2\left(\frac{2p}{n}K(k_s),k_s\right)}}\,; \tag{10.148}$$

$$\text{n = even:}\quad R_n(y,L) = C_2 \prod_{p=1}^{n/2} \frac{y^2 - sn^2\left(\frac{2p-1}{n}K(k_s),k_s\right)}{y^2 - \dfrac{y_s^2}{sn^2\left(\frac{2p-1}{n}K(k_s),k_s\right)}}\,, \tag{10.149}$$

as in Eqs. (10.81) and (10.82). The zeros of $R_n(y,L)$ are

$$\text{n = odd :}\quad y_{z,p} = sn\left(\frac{2p}{n}K(k_s),k_s\right), \qquad p = 1,\ldots,(n-1)/2\,. \tag{10.150}$$

$$\text{n = even :}\quad y_{z,p} = sn\left(\frac{2p-1}{n}K(k_s),k_s\right), \qquad p = 1,\ldots,n/2\,. \tag{10.151}$$

Associated with each zero, there is pole:

$$y_{q,p} = \frac{y_s}{y_{z,p}}\,. \tag{10.152}$$

The constants C_1 and C_2 are given by

$$C_1 = \prod_{p=1}^{(n-1)/2} \frac{1-y_{q,p}^2}{1-y_{z,p}^2}\,, \quad \text{n = odd}\,. \tag{10.153}$$

$$C_2 = \prod_{p=1}^{n/2} \frac{1-y_{q,p}^2}{1-y_{z,p}^2}\,, \quad \text{n = even}\,. \tag{10.154}$$

10.5 Remarks

We have presented the design methodology of three types of low-pass filters: Butterworth, Chebyshev, and Cauer. Given the specifications of a filter, namely, the passband edge, stop-band edge, maximum insertion loss in the passband, minimum insertion loss in the stop-band, and the terminating resistances R_1 and R_2, we can find the minimum order of the filter and proceed to synthesize it algorithmically. The whole design process from specifications to determining the circuit configuration and element values can indeed be automated and computer programs for this purpose are available [72]. Explicit formulas for the element values of the three types of filters exist [17, 56, 59, 68, 72], and design tables as well [59, 62, 68, 73]. Nowadays mathematical programming tools are widely available and filter design can easily be done step by step on a personal computer. There is no need to remember any formula or to store any table.

The design methodology is based on the realization of the given $|S_{21}(j\omega)|^2$ as the transmission power gain of a lossless ladder terminated in R_1 and R_2. There may be other realizations, but the ladder seems most efficient in that the number of ladder branches equals the order of the transmission function $S_{21}(s)$.

The three types of low-pass filters belong to a class of filters whose transmission power gain is characterized by:

$$|S_{21}(j\omega)|^2 = \frac{A}{1+\varepsilon^2\Phi^2(\omega)}, \tag{10.155}$$

where Φ is some real function of ω, often called the characteristic function because it determines the frequency characteristics of the filter. For Butterworth filters, $\Phi(\omega) = \omega^n$, a polynomial of one term. For Chebyshev filters, $\Phi(\omega) = T_n(\omega)$, a Chebyshev polynomial. For Cauer filters, $\Phi(\omega) = R_n(\omega)$, a rational function whose zeros are elliptic sine functions of the parameters and whose poles are reciprocally related to the zeros. A question arises as to whether or not there exist other polynomials or rational functions which will yield better filter performance for the same order n.

10.5.1 Cauer filters

We recall that a low-pass filter is required to have a loss characteristic:

$$\alpha_{dB}(\omega) = -10\log|S_{21}(j\omega)|^2 \tag{10.156}$$

that approximates zero or a small constant in the passband, and a large constant in the stop-band. The Cauer filter meets these requirements optimally in the sense that in the passband the maximum deviation from a small constant is minimized (equal-ripple) and that in the stop-band, the maximum deviation from a large constant is also minimized (equal-ripple). By choosing $\Phi(\omega)$ to be a rational function, we place the transmission zeros (loss poles) at finite frequencies closer to the stop-band edge. This allows us to have a very steep cut-off characteristics. In the passband, we create as many points of zero loss from a constant as the order allows and place them in such a way that the ripples are all equal. Cauer filters are the preferred choice in all applications that require a sharp transition from passband to stop-band.

10.5.2 Chebyshev filters

The Chebyshev filter is optimum in the sense that in the passband, the loss approximates a constant in an equal-ripple manner with as many points of zero loss from a constant as the order allows and in the stop-band it goes to infinity monotonically. It has been shown [57] that no other polynomial of the same order has these properties.

It is instructive to compare the loss characteristics of a Chebyshev filter with those of a Butterworth filter in the stop-band. The Chebyshev polynomial can be written as [36]:

$$T_n(\omega) = \omega^n - \binom{n}{2}\omega^{n-2}(1-y^2) + \binom{n}{4}\omega^{n-4}(1-\omega^2)^2 - \binom{n}{6}\omega^{n-6}(1-\omega^2)^3 + \cdots \tag{10.157}$$

$$T_n(\omega)|_{\omega\to\infty} = \left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots\right]\omega^n = 2^{n-1}\omega^n\,. \tag{10.158}$$

At high frequencies, the transmission loss of a Chebyshev filter is:

$$\alpha_{dB}(\omega) = -10\log A + 10\log\varepsilon^2 + 10(2n-2)\log 2 + 10\log\omega^{2n}\,. \tag{10.159}$$

On the other hand, for a Butterworth filter of the same order, we have:

$$\alpha_{dB}(\omega) = -10\log A + 10\log\varepsilon^2 + 10\log\omega^{2n}\,. \tag{10.160}$$

The Chebyshev filter is superior to the Butterworth filter by the third term, which for $n = 5$, for example, is 24 dB.

10.6 Loss sensitivity of filters

In both the Chebyshev and Cauer filters, we choose a characteristic function $\Phi(\omega)$ such that the transmission loss from a constant is zero at finite frequencies in the passband. A consequence of this property is that the sensitivity of the loss with respect to a small variation of any element of the filter is zero at these frequencies and it is small overall in the passband.

10.6.1 Passband sensitivity

Let the transmission loss (or loss for short) be

$$\alpha(\omega) = -\ln|S_{21}(j\omega)|\,. \tag{10.161}$$

We recall that $|S_{21}(j\omega)| \le 1$ in an *RLC* circuit. So $\alpha(\omega) \ge 0$ for all ω no matter what the element values of the filter are. Let α_m be the minimum value of $\alpha(\omega)$ in the passband. For Chebyshev and Cauer filters, the minima of $\alpha(\omega)$ are all equal to α_m. Let the minima be at ω_k, $k = 1, \ldots$. Let

$$\alpha'(\omega) = \alpha(\omega) - \alpha_m \ge 0\,. \tag{10.162}$$

Then $\alpha'(\omega_k) = 0$. Let x be any element value. Consider the loss when x is changed to $x + \Delta x$. Expand $\alpha'(\omega, x)$ about x in Taylor series:

$$\alpha'(\omega, x+\Delta x) = \alpha'(\omega, x) + \Delta x \frac{\partial \alpha'(\omega, x)}{\partial x} \geq 0, \tag{10.163}$$

At a minimum ω_k:

$$\alpha'(\omega_k, x+\Delta x) = \Delta x \frac{\partial \alpha(\omega_k, x)}{\partial x} \geq 0. \tag{10.164}$$

But Δx may be positive or negative and we conclude that condition (10.164) can only be satisfied if:

$$\frac{\partial \alpha(\omega_k, x)}{\partial x} = 0. \tag{10.165}$$

That is to say, the sensitivity of the loss function $\alpha(\omega)$ with respect to a small change in any element value is zero at its minima in the passband. Since the loss is small and varies smoothly across the passband in a well-designed filter, we conclude that the sensitivity throughout the passband is small. This observation was made by Alfred Fettweiss in 1956 [45, 25] and independently by Orchard ten years later [48].

10.6.2 Loss sensitivity bounds

The foregoing is an approximate analysis. To compute the loss sensitivity, we may invoke the formulas given in Eqs. (4.85) and (4.96). However, a more meaningful quantity is its bounds as a function of frequency. For this purpose, an analytic expression of the loss sensitivity is necessary. Loss sensitivity can be expressed in many different but equivalent ways [38, 7, 64, 49]. The following derivation is based on that reported in [64].

Consider a filter circuit shown in Fig. 10.18. The two-port consists of inductors and capacitors only and hence is lossless. Let the transmission function be $S_{21}(s)$. We recall that

$$|S_{21}(j\omega)|^2 = 4\frac{R_1}{R_2}\left|\frac{V_2}{E}\right|^2 = \frac{8R_1}{E^2}P_2. \tag{10.166}$$

where $P_2 = |V_2|^2/2R_2$ is the average power delivered to R_2. Since the two-port is lossless, P_2 is equal to the power taken by the circuit to the right of port-1, P_1, which is

$$P_1 = \frac{1}{2}\frac{E^2\Re Z_1}{|R_1+Z_1|^2}. \tag{10.167}$$

Combining equations, we get

$$|S_{21}(j\omega)|^2 = \frac{4R_1\Re Z_1}{|R_1+Z_1|^2}. \tag{10.168}$$

Consider the sensitivity of $|S_{21}(j\omega)|^2$ with respect to an inductance L_k first. We have

$$\frac{\mathrm{d}\,|S_{21}(j\omega)|^2}{\mathrm{d}L_k} = 4R_1 \frac{\mathrm{d}}{\mathrm{d}L_k}\left[\frac{\Re Z_1}{|R_1+Z_1|^2}\right] = \frac{4R_1}{|R_1+Z_1|^2}\Re\left[\frac{R_1-Z_1^*}{R_1+Z_1}\frac{\mathrm{d}Z_1}{\mathrm{d}L_k}\right]. \quad (10.169)$$

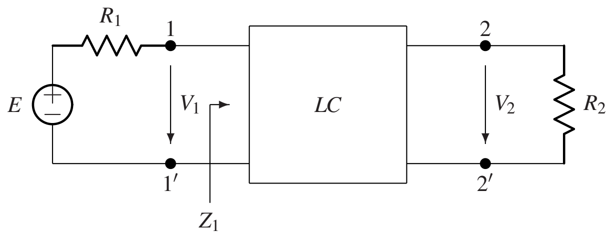


Fig. 10.18 A lossless two-port terminated in R_1 and R_2 at port-1 and port-2, respectively

It remains to find $\mathrm{d}Z_1/\mathrm{d}L_k$. Using the current at port-1 as an input source, applying formula (4.95) to V_1 and $y = 1/(j\omega L_k)$, and noting the admittance matrix is symmetrical, we get

$$\frac{\mathrm{d}Z_1}{\mathrm{d}L_k} = j\omega\frac{I_{L_k}^2}{I_1^2}, \quad (10.170)$$

where I_{L_k} is the current in the inductor L_k. Substituting Eq. (10.170) into Eq. (10.169) and noting that $I_1 = E/(R_1+Z_1)$, we have

$$\frac{\mathrm{d}\,|S_{21}(j\omega)|^2}{\mathrm{d}L_k} = -\frac{4R_1}{E^2}\Re\left[S_{11}^*(j\omega)\,j\omega I_{L_k}^2\right]. \quad (10.171)$$

Multiplying the equation by L_k, dividing by $|S_{21}(j\omega)|^2$, and using Eqs. (10.166) and (10.161), we obtain finally the loss sensitivity with respect to L_k given by:

$$L_k\frac{\mathrm{d}\alpha}{\mathrm{d}L_k} = \frac{\omega}{P_2}\Im\left[S_{11}^*(j\omega)\frac{1}{4}LI_{L_k}^2\right] \quad (10.172)$$

Let $S_{11}(j\omega) = |S_{11}(j\omega)|\,\mathrm{e}^{j\theta_{11}}$ and $I_{L_k} = \left|I_{L_k}\right|\mathrm{e}^{j\phi_L}$. Then we have

$$L_k\frac{\mathrm{d}\alpha}{\mathrm{d}L_k} = \frac{\omega}{P_2}\,|S_{11}(j\omega)|\,W_L\sin(2\phi_L-\theta_{11}) \quad (10.173)$$

where W_L is the average energy stored in L_k, and the bound is:

$$\left|L_k\frac{\mathrm{d}\alpha}{\mathrm{d}L_k}\right| \le \frac{\omega}{P_2}W_L\sqrt{1-\mathrm{e}^{-2\alpha}}, \quad (10.174)$$

in which we have used the unitary properties of a lossless two-port.

In a similar manner, we obtain the loss sensitivity with respect to a capacitance C_k, given by

$$C_k \frac{\mathrm{d}\alpha}{\mathrm{d}C_k} = -\frac{\omega}{P_2} |S_{11}(j\omega)| W_C \sin(2\phi_C - \theta_{11}), \tag{10.175}$$

where W_C is the average energy stored in C_k and ϕ_C is the argument of the capacitor voltage. The bound is found to be

$$\left| C_k \frac{\mathrm{d}\alpha}{\mathrm{d}C_k} \right| \leq \frac{\omega}{P_2} W_C \sqrt{1 - \mathrm{e}^{-2\alpha}}. \tag{10.176}$$

If there are more than one inductor and one capacitor, the worst case bound is the sum of the bounds with respect to all the inductances and all the capacitances.

The loss is also sensitive to changes in the terminating resistances R_1 and R_2. Since R_2 is part of the circuit to the right of port-1, the loss sensitivity with respect to it is obtained in the same way as that with respect to L_k or C_k, to wit,

$$R_2 \frac{\mathrm{d}\alpha}{\mathrm{d}R_2} = -\frac{1}{2} |S_{11}(j\omega)| \cos(2\phi_{R_2} - \theta_{11}), \tag{10.177}$$

where ϕ_{R_2} is the argument of the current in R_2. The sensitivity bound is

$$\left| R_2 \frac{\mathrm{d}\alpha}{\mathrm{d}R_2} \right| \leq \frac{1}{2} \sqrt{1 - \mathrm{e}^{-2\alpha}}. \tag{10.178}$$

To find the loss sensitivity with respect to R_1, it is best to invoke reciprocity and we obtain

$$R_1 \frac{\mathrm{d}\alpha}{\mathrm{d}R_1} = -\frac{1}{2} |S_{22}(j\omega)| \cos(2\phi_{R_1} - \theta_{22}), \tag{10.179}$$

where $S_{22}(j\omega)$ is the reflection coefficient at port-2 and θ_{22} its argument, and ϕ_{R_1} is the argument of the current in R_1. The bound is identical to that with respect to R_2 owing to the unitary properties and is given by

$$\left| R_1 \frac{\mathrm{d}\alpha}{\mathrm{d}R_1} \right| \leq \frac{1}{2} \sqrt{1 - \mathrm{e}^{-2\alpha}}. \tag{10.180}$$

From the expressions for the sensitivity bounds, it is clear that at frequencies where the transmission loss is zero, the loss sensitivity with respect to any element value is zero. In the passband where the loss is nearly constant, the sensitivity with respect to R_1 and R_2 is nearly independent of frequency, namely, a small change in either terminating resistance will cause the loss characteristics to shift up or down by a small amount uniformly over the passband.

10.6.3 Example

As an example, consider the fifth order Cauer low-pass filter of Fig. 10.11. Its minimum loss in the passband $\alpha_{min} = 0$. Fig. 10.19 shows the loss in solid line when all the element values assume their designed values. The dotted line is the passband loss when the value of C_2 is increased by 2% and the dashed line is the loss when it is decreased by 2%. It is evident that as the value of C_2 changes, the loss remains small and greater than zero throughout the passband, and it retains the general form of the designed loss. Moreover, the zero-loss frequencies are relatively stationary. Similar results are obtained for changes in the inductances and other capacitances. A small change in the terminating resistances will result in a small shift of the loss characteristic but the general shape remains unchanged (see Problem 10.9).

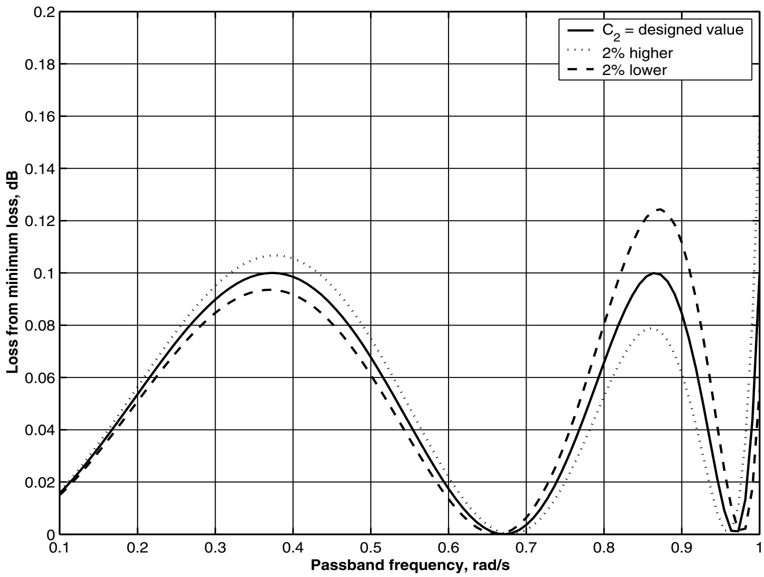


Fig. 10.19 Passband loss of a fifth order Cauer filter showing the effect of a 2% change in the middle capacitor on the loss

The key to these important results is that in a passive circuit $|S_{21}(j\omega)|^2 \leq 1$ so that $\alpha(\omega) \geq 0$ for all ω and all element values. It follows that it does not matter how $S_{21}(s)$ is realized in a circuit. As long as the circuit has the property that its transmission power gain $|S_{21}(j\omega)|^2 \leq 1$, the transmission loss at the passband zeros will be insensitive to small element variations. The lossless two-port which we use to realize $S_{21}(s)$ has this property. For this reason, many, if not most, analog electronic filters simulate the lossless ladder configuration element by element, by means of integrators (to simulate the capacitors and inductors) and summers (to simulate *KCL* and *KVL*) [61, 69], and this configuration will have small loss sensitivities to element variations.

10.7 Analog computer simulation of filters

We need two kinds of elements: a scaled summer and a scaled integrator depicted in Fig. 10.20.

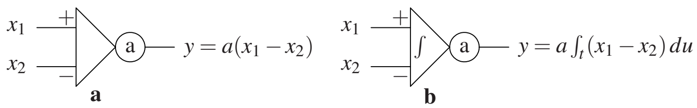


Fig. 10.20 **a** scaled summer and **b** scaled integrator for analog computer simulation of filters

Assume we have designed a low-pass filter in the form of a lossless ladder terminated in R_1 and R_2 as shown in Fig. 10.21. With an assignment of node voltages and element currents as indicated, the circuit equations are:

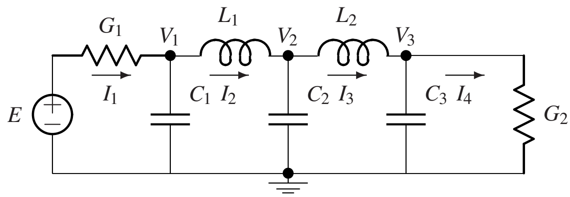


Fig. 10.21 Low-pass filter for analog computer simulation

$$
\begin{aligned}
I_1 &= G_1(E - V_1)\,, & V_1 &= \frac{1}{C_1}\int_t (I_1 - I_2)\,\mathrm{d}u\,, & I_2 &= \frac{1}{L_1}\int_t (V_1 - V_2)\,\mathrm{d}u\,,\\
V_2 &= \frac{1}{C_2}\int_t (I_2 - I_3)\,\mathrm{d}u\,, & I_3 &= \frac{1}{L_2}\int_t (V_2 - V_3)\,\mathrm{d}u\,, & V_3 &= \frac{1}{C_3}\int_t (I_3 - I_4)\,\mathrm{d}u\,,\\
I_4 &= G_2V_3\,.
\end{aligned}
$$

These equations are implemented in an analog computer ”circuit” directly, element by element, as shown in Fig. 10.22. If the summers and integrators are ideal, this circuit has the same property as the passive filter and the sensitivity at the zero loss frequencies in the passband is zero.

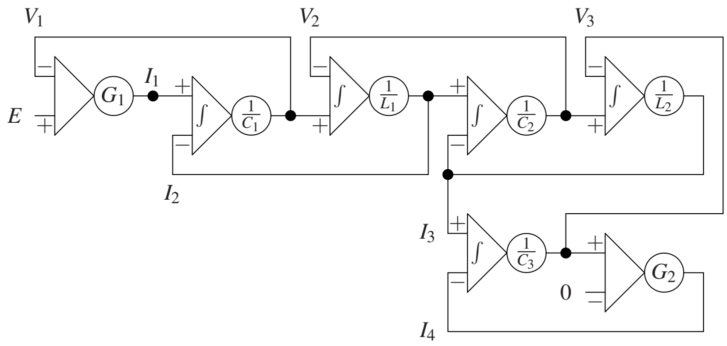


Fig. 10.22 Analog computer simulation of a fifth degree low-pass filter

10.8 Frequency transformation

High-pass, band-pass, and band-elimination filters can all be derived from a low-pass filter by frequency transformation. The transformation is one which replaces the complex frequency s by a positive real function, so that the transformed impedances are all positive real and the transformed circuit is again passive. So if a transfer function in the s domain is bounded real, for example, the transfer function in the transformed circuit is also bounded real. In fact, if we choose a transformation such that the numerical value of each impedance at a frequency is equal to the numerical value of the transformed impedance at the corresponding transformed frequency, then the transfer function of the transformed circuit will have the same frequency characteristics as the original transfer function on a transformed frequency scale. So frequency transformation amounts to replacing each impedance by another according to the positive real function which replaces s.

10.8.1 Low-pass to high-pass transformation

In a high-pass (HP) filter, the passband extends from some reference frequency to infinity and the stop-band covers the range from zero to the reference frequency. In other words, the passband of a low-pass (LP) filter becomes the stop-band of a HP filter, and vice versa. This suggests a transformation:

$$s = \frac{\omega_p^2}{p} \tag{10.181}$$

where $s = \sigma + j\omega$ is the complex frequency variable in the low-pass (LP) domain and $p = \Sigma + j\Omega$ is the frequency variable in the HP domain. The mapping given in Eq. (10.181) transforms the frequency range $[0, \pm\omega_p]$ in the s-domain to the range

$[\mp\infty, \mp\omega_p]$ in the p-domain. If ω_p is the passband edge, the transformation takes a LP filter into a HP filter, as we desire. Moreover, if we make the following replacements of the elements, then a LP transfer function $F(s)$ becomes a HP transfer function $F'(p) = F(\omega_p^2/p)$ and they have the same frequency characteristics in their respective passbands and stop-bands.

LP element	LP impedance	HP impedance	HP element
R	R	R	$R' = R$
L	sL	$\omega_p^2 L/p$	$C' = 1/\omega_p^2 L$
C	$1/sC$	$p/\omega_p^2 C$	$L' = 1/\omega_p^2 C$

Example 10.1. Let us transform the Chebyshev low-pass filter of Fig. 10.7 into a Chebyshev high-pass filter by the transformation of Eq. (10.181) with the passband edge at 80 KHz. The filter is shown in Fig. 10.23, and the element values are given below.

LP element	LP value	HP element	HP value
C_1	1.597591 nF	L_1'	2.477392 mH
L_2	2.807926 mH	C_2'	1.409531 nF
C_3	3.519709 nF	L_3'	1.124485 mH
L_4	3.259893 mH	C_4'	1.214107 nF
C_5	2.947657 nF	L_5'	1.342713 nH

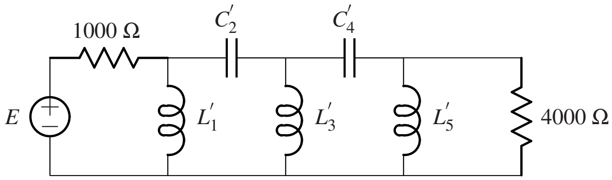


Fig. 10.23 A Chebyshev high-pass filter derived from a low-pass filter by frequency transformation. See Fig. 10.7

Its insertion loss is shown in Fig. 10.24. It is seen the equi-ripple characteristics are preserved in the HP passband, as expected.

10.8.2 Low-pass to band-pass transformation

We look for a transformation which will take the entire passband extending from $-\omega_p$ to ω_p of a LP filter and map it into a passband $[\Omega_1, \Omega_2]$, and by symmetry, $[-\Omega_1, -\Omega_2]$ as well. Let $s = \sigma + j\omega$ be the frequency variable in the LP domain and $p = \Sigma + j\Omega$ be that in the BP domain. The transformation is:

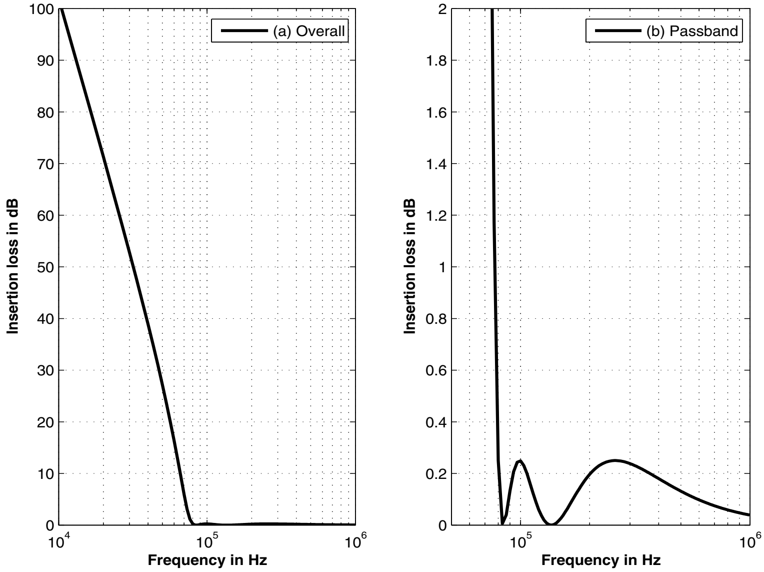


Fig. 10.24 Insertion loss of a fifth order Chebyshev high-pass filter

$$s = \frac{p^2 + \Omega_0^2}{Bp}, \tag{10.182}$$

where B is a frequency scaling factor. Note that the right-hand side is a positive real function, so the impedances in the BP domain are all positive real. The transformation maps the origin of the s domain into the frequencies $\pm j\Omega_0$. Each LP frequency pair $\pm\omega$ are mapped into four BP frequencies:

$$\Omega_2^+ = \frac{B\omega}{2} + \frac{1}{2}\sqrt{B^2\omega^2 + 4\Omega_0^2} > 0, \tag{10.183}$$

$$\Omega_1^+ = -\frac{B\omega}{2} + \frac{1}{2}\sqrt{B^2\omega^2 + 4\Omega_0^2} > 0, \tag{10.184}$$

$$\Omega_2^- = -\frac{B\omega}{2} - \frac{1}{2}\sqrt{B^2\omega^2 + 4\Omega_0^2} < 0, \tag{10.185}$$

$$\Omega_1^- = \frac{B\omega}{2} - \frac{1}{2}\sqrt{B^2\omega^2 + 4\Omega_0^2} < 0. \tag{10.186}$$

Note that

$$\Omega_1^{\pm}\Omega_2^{\pm} = \Omega_0^2 \tag{10.187}$$

for every $\pm\omega$ pair, so that $\Omega_1^{\pm}$ and $\Omega_2^{\pm}$ are geometrically symmetrical with respect to Ω_0. It follows that the BP frequency characteristics are geometrically symmetrical with respect to Ω_0. The passband edges $\pm\omega_p$ in the LP filter are mapped into BP passband edges (positive frequencies, similarly for negative frequencies) given by:

$$\Omega_{2p}^{+} = \frac{B\omega_p}{2} + \frac{1}{2}\sqrt{B^2\omega_p^2 + 4\Omega_0^2} > 0\,, \tag{10.188}$$

$$\Omega_{1p}^{+} = -\frac{B\omega_p}{2} + \frac{1}{2}\sqrt{B^2\omega_p^2 + 4\Omega_0^2} > 0\,. \tag{10.189}$$

The BP passband becomes:

$$\Omega_{2p}^{+} - \Omega_{1p}^{+} = B\omega_p\,. \tag{10.190}$$

The transformation of Eq. (10.182) requires that an inductor in the LP filter be replaced by an inductor in series with a capacitor in the BP filter, and a capacitor replaced by a capacitor in parallel with an inductor. The element values are computed as shown below:

LP element	BP impedance	BP element
R	R'	$R' = R$
L	$pL' + \dfrac{1}{pC'}$	$L' = \dfrac{L}{B},\ C' = \dfrac{B}{\Omega_0^2 L}$
C	$\dfrac{1}{pC'' + \dfrac{1}{pL''}}$	$C'' = \dfrac{C}{B},\ L'' = \dfrac{B}{\Omega_0^2 C}$

Example 10.2. Let us transform the Chebyshev low-pass filter of Fig. 10.7 into a band-pass filter with passband edges at 890 MHz and 910 MHz. We need the following parameters:

$$\Omega_0 = 2\pi\sqrt{890 \times 910} = 5.6545177 \times 10^9 \text{rad/s}\,. \tag{10.191}$$

$$B = \frac{\Omega_2 - \Omega_1}{\omega_p} = 250\,. \tag{10.192}$$

The transformation is now determined. The low-pass stop-band edges are transformed into band-pass stopband edges at ± 863.225 MHz and ± 938.225 MHz. Over the passband and stop-band, the insertion loss characteristics are the same as those of the low-pass filter in its passband and stop-band.

The band-pass filter is shown in Fig. 10.25 and its element values are given in the table below. Fig. 10.26 shows the insertion loss characteristics.

LP element value	BP capacitance	BP inductance
$C_1 = 1.597591$ nF	$C_1' = 6.390364$ pF	$L_1' = 4.894217$ nH
$L_2 = 2.807926$ mH	$C_2' = 2.784602$ fF	$L_2' = 11.231704$ µH
$C_3 = 3.519709$ nF	$C_3' = 14.078836$ pF	$L_3' = 2.221478$ nH
$L_4 = 3.259893$ mH	$C_4' = 2.398532$ fF	$L_4' = 13.039572$ µH
$C_5 = 2.947657$ nF	$C_5' = 11.790628$ pF	$L_5' = 2.652600$ nH

It must be said that in actual design, we specify the band-pass filter characteristics first, translate the specifications to LP specifications, design the LP filter and obtain

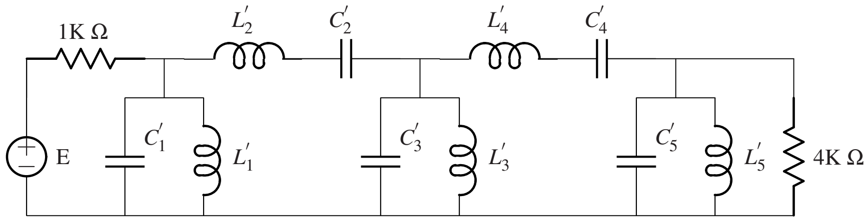


Fig. 10.25 A band-pass filter derived from the low-pass filter of Fig. 10.7

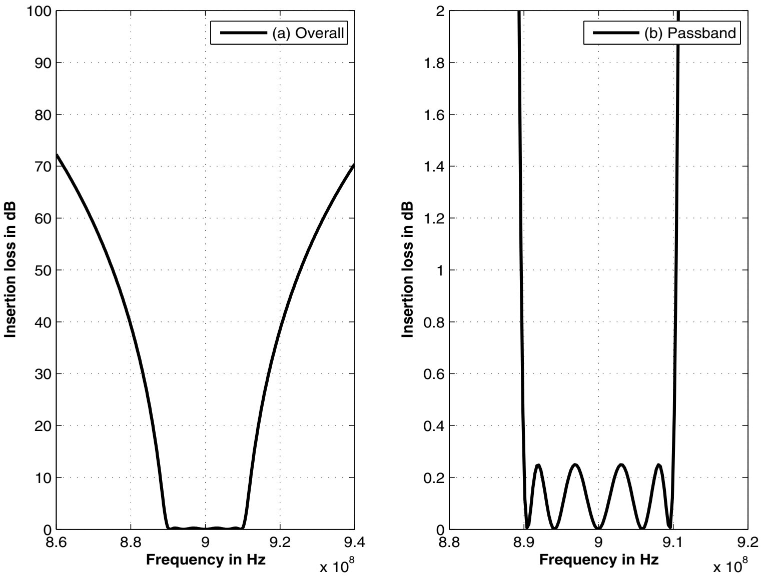


Fig. 10.26 Insertion loss of a fifth order Chebyshev band-pass filter

the BP filter by replacement of elements. The BP specifications include the passband edges Ω_{1p} and Ω_{2p}, maximum insertion loss in the passband, one of the stop-band edges (the other being geometrically symmetrical with respect to Ω_0), and minimum insertion loss in the stop-band. The frequency scaling factor B is arbitrary and can be chosen to make $\omega_p = 1$ rad/s for example.

10.8.3 Low-pass to band-elimination transformation

A band-elimination (BE) filter is the inverse of a band-pass filter. The passband of one is the stop-band of the other and vice versa. The transformation from a low-pass to a band-elimination filter is given by:

$$s = \frac{B\omega_p^2 p}{p^2 + \Omega_0^2} \tag{10.193}$$

The elements are replaced as follows.

LP element	BE element
R	$R' = R$
L	$L' = \dfrac{B\omega_p^2 L}{\Omega_0^2}$ in parallel with $C' = \dfrac{1}{B\omega_p^2 L}$
C	$C'' = \dfrac{B\omega_p^2 C}{\Omega_0^2}$ in series with $L'' = \dfrac{1}{B\omega_p^2 C}$

Example 10.3. Let us transform the Chebyshev low-pass filter of Fig. 10.7 into a band-elimination filter with passband edges at 890 MHz and 910 MHz. We will omit all the details. The filter is shown in Fig. 10.27 and its element values are given in the table below. Fig. 10.28 shows the insertion loss characteristics.

LP element value	BE capacitance	BE inductance
$C_1 = 1.597591$ nF	$C_1' = 3.156125$ fF	$L_1' = 9.909567$ µH
$L_2 = 2.807926$ mH	$C_2' = 5.638124$ pF	$L_2' = 5.547205$ nH
$C_3 = 3.519709$ nF	$C_3' = 6.953370$ fF	$L_3' = 4.497938$ µH
$L_4 = 3.259893$ mH	$C_4' = 4.856428$ pF	$L_4' = 6.440090$ nH
$C_5 = 2.947657$ nF	$C_5' = 5.823251$ fF	$L_5' = 5.370854$ µH

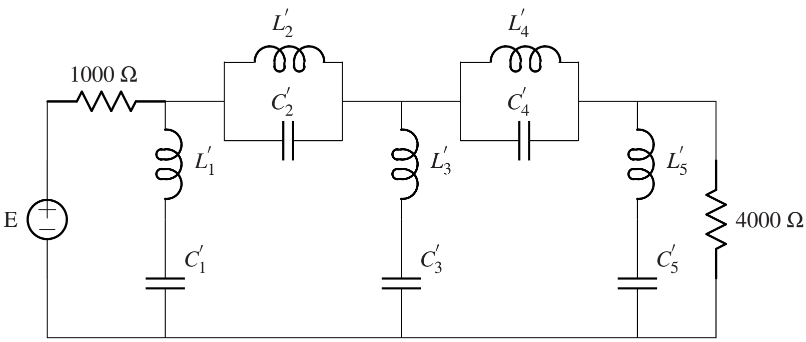


Fig. 10.27 A band-elimination filter derived from a low-pass filter

Problems

10.1. In many a digital communication system, “quadrature amplitude modulation” [39] is often used for two base-band signals $g_1(t)$ and $g_2(t)$ to share the same chan-

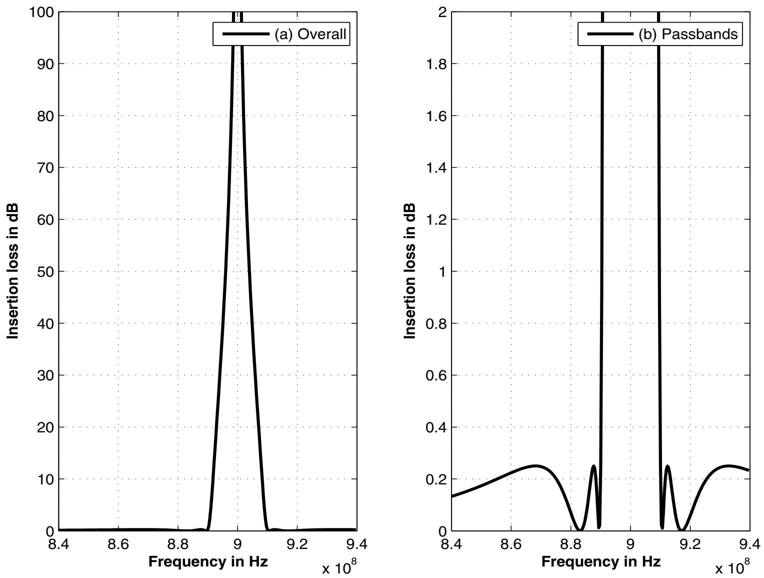


Fig. 10.28 Insertion loss of a fifth order Chebyshev band-elimination filter

nel in order to conserve bandwidth. The two signals are modulated by two carrier signals in quadrature, $\sin\omega_c t$ and $\cos\omega_c t$, respectively. The modulated signals are combined and transmitted. At the receiver, the combined signal is demodulated by two quadrature carrier signals which are in phase with the carrier signals in the transmitter. The two demodulated signals are then filtered to remove all high frequency components, and $g_1(t)$ and $g_2(t)$ are recovered. Design two Butterworth low-pass filters for this purpose. Let $\omega_c = 600\,\text{MHz}$. All high frequency components should be suppressed by at least 60 dB.

10.2. A waveform $g(t)$ can be generated by a sequence of narrow rectangular pulses $p(t-nT)$, $n = 0, 1, \ldots$, whose amplitudes are proportional to the values of the waveform at $t = nT$. The resultant signal will be jagged and need be smoothed by removing the high frequency components. Consider generating a sine-wave of frequency 1 KHz with twenty pulses per period. Design a Butterworth low-pass filter to suppress the second harmonic by at least 60 dB. Plot the smoothed output.

10.3. Design a Butterworth low pass filter with the following specifications: $R_1 = 50\Omega$ and $R_2 = 10\Omega$, maximum passband loss $\alpha_{max} = 0.25\,\text{dB}$, passband edge $\omega_p = 56\,\text{KRad/s}$, minimum insertion loss $\alpha_{min} = 40\,\text{dB}$ at $\omega_s = 144\,\text{KRad/s}$. Compute the group delay of the filter. Plot the frequency response.

10.4. Repeat Problems 10.1 and 10.3 with a Chebyshev filter.

10.5. An inverse Chebyshev low-pass filter is one which has a maximally flat passband and an equi-ripple stop-band. Show that the following transmission power gain has these characteristics:

$$|S_{21}(\omega)|^2 = A\frac{\varepsilon^2 T_n^2(1/\omega)}{1+\varepsilon^2 T_n^2(1/\omega)}.$$

Let ω_p and ω_s be the passband and stop-band edges, respectively. Let α_{max} be the maximum insertion loss in the passband. Let α_{min} be the minimum insertion loss in the stop-band. Derive a formula that obtains the order of the filter to satisfy these requirements.

10.6. Design an elliptic low-pass filter with a passband ripple of 0.5 dB, passband edge at 900 MHz, a minimum stop-band insertion loss of 60 dB, and a stop-band edge at 1800 MHz. The filter is terminated at the input end in 50 Ω and in 10 Ω at the output end. What is the order of the filter if Butterworth or Chebyshev approximation is used?

10.7. Provide all the missing steps in the derivation of the loss sensitivity with respect to a percentage change in an inductance, from Eq. (10.166) to (10.173).

10.8. Derive all the loss sensitivity bounds of Sect. 10.6.2.

10.9. In the Cauer filter of Fig. 10.11, study the effect of element changes on the passband loss. Change the inductances and capacitances by $\pm 2\%$ one at a time and all at a time. Also demonstrate that the effects of changes of the terminating resistances are to shift the level of the passband characteristics while the general form does not change.

10.10. Study the sensitivity of the passband loss with respect to element changes in a Chebyshev low-pass filter. Experiment with a fifth order filter for different passband ripples.

10.11. Design a maximally flat bandpass filter to be inserted between a power amplifier and an antenna for use in a cellular phone system. Assume the amplifier has an output impedance of 5 Ω and the antenna presents a load of 50 Ω to the filter. The filter's 1-dB passband edges are at 864 MHz and 894 MHz (base cell station transmit band). It is to have an insertion loss of 40 dB at the center of the next transmit band at 836 MHz. Plot the response of the filter. Next, let the Q of all the inductors be 100 at the center of the passband. Plot the response. Repeat for $Q = 20$.

10.12. Repeat the last problem with an elliptic band-pass filter except that the passband ripple is changed to 0.5 dB and the minimum insertion loss in the stop-band is increased to 80 dB. What is the order of the filter if Butterworth or Chebyshev design is used?

10.13. The FM broadcast band extends from 88 MHz to 108 MHz. Design a bandpass filter with its passband edges at these frequencies. The maximum loss in the passband is 0.2 dB and the minimum loss in the stop-band is 60 dB, with an upper stop-band edge at 120 MHz. Compare the order of the filter if a Butterworth, Chebyshev, or Cauer design is used. Realize the filter in the Cauer design. Let the terminating resistances at the ends be $50\,\Omega$. What will be the orders of the three filters if we change the stop-band edge to $113\,\mathsf{MHz}$?

10.14. In color television, the video signal is decomposed into several components. One is a signal to control the "color level" and it is extracted from the input signal through a band-pass filter with a pass-band from $2\,\mathsf{MHz}$ to $4.2\,\mathsf{MHz}$. Design a Cauer filter with a maximum loss in the pass-band of 0.1 dB and a minimum loss at $4.5\,\mathsf{MHz}$ of 40 dB.

10.15. A telephone channel filter is a band-pass filter with a bandwidth of $8\,\mathsf{KHz}$ centered at $100\,\mathsf{KHz}$, with a passband ripple of 0.5 dB. The insertion loss at the center of the next channel at $120\,\mathsf{KHz}$ should be at least 40 dB. Design three filters, each of which will be inserted between 50 ohm resistors at the input and output ends, with a Butterworth, Chebyshev, and Cauer characteristic, respectively.

10.16. Let $D(\omega)$ be the group delay of a low-pass filter. Apply a low-pass to band-pass transformation to it to obtain a bandpass filter. What is the group delay of the band-pass filter?

10.17. Obtain an analog computer implementation of a fifth order Cauer low-pass filter. What is the order of the state equations? How many integrators do you need?

10.18. Obtain an analog computer implementation of a third order band-pass filter of the Cauer design.

10.19. Obtain an analog computer implementation of a third order high-pass filter of the Butterworth design.

10.20. Filters are frequency selective circuits in that a filter suppresses the passage of signals in certain frequency range and lets pass signals in another frequency range. There is another class of frequency selective circuits known as gain or loss equalizers which when cascaded with another circuit correct the frequency response of the latter so that the overall response will have some desired characteristic. If the multiplicative constant of a transfer function is not important, these equalizers can also be used to realize certain types of filters. The most common type of loss equalizers is the constant-resistance bridged-T two-port inserted between two equal resistances as shown in Fig. 10.29, with the condition that $Z_aZ_b = R^2$ for all frequencies.

1. Show that the transfer function is given by:

$$H(s) = \frac{V_{out}}{E} = \frac{1}{2\left(1+\dfrac{Z_a}{R}\right)}.$$

   Thus the frequency response is controlled entirely by the impedance Z_a.
2. Show that $H(s)$ is a positive real function.

3. What must Z_a and Z_b be if the transfer function is to have a simple low-pass, high-pass, band-pass, or band-elimination characteristic?
4. Show that the input impedance across $1-1'$ is R, independent of frequency.
5. Show that when n constant-resistance bridged-T two-ports are connected in cascade and inserted between equal resistances R, the overall transfer function is:

$$H(s) = \frac{1}{2}\prod_{k=1}^{n}\frac{1}{1+\dfrac{Z_{ak}}{R}}$$

where Z_{ak} is the Z_a of the kth bridged-T two-port. Thus, if a transfer function can be factored into a product of rational functions, it can be synthesized as a cascade of bridged-T two-ports each of which realizes one of the factored rational functions.

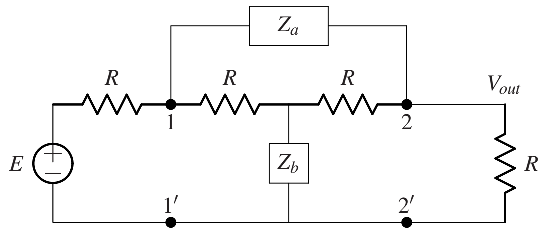


Fig. 10.29 Constant-resistance bridged-T two-port inserted between two equal resistances R, with $Z_aZ_b = R^2$

10.21. Use the circuit of Problem 10.20, consisting of a cascade of two constant-resistance bridged-T two-ports, to realize each of the following transfer functions, with a suitable multiplicative constant K in each case.

$$(a)\ |H(j\omega)|^2 = \frac{K\omega^4}{1+\omega^6}. \qquad (b)\ |H(j\omega)|^2 = \frac{K\omega^2}{1+\omega^6}.$$

$$(c)\ |H(j\omega)|^2 = \frac{K\omega^4}{4\omega^6+9\omega^4+6\omega^2+1}. \qquad (d)\ |H(j\omega)|^2 = \frac{K\omega^2}{4\omega^6+9\omega^4+6\omega^2+1}.$$

10.22. With reference to Problem 10.20, (a) what are the necessary and sufficient conditions on a rational function to be realizable as the transfer function of a constant-resistance bridged-T circuit? (b) What are the necessary and sufficient conditions on a rational function to be realizable as the transfer function of a constant-resistance bridged-T circuit in which Z_a is an LC impedance function?

Chapter 11
Circuit Design by Optimization

Butterworth, Chebyshev, and Cauer low-pass filters are examples of filters whose gain characteristic approximates that of an ideal low-pass filter. We saw in the last chapter that their design can be obtained algorithmically. Indeed, computer programs exist that will generate the element values of a design for any degree of selectivity[72]. There are situations, however, in which the gain response is not rectangular but is instead some special function of frequency. For example, the vestigial filter used in TV transmitters and receivers has a gain response that drops off rapidly at frequencies near the upper sideband but gradually at frequencies near the lower sideband. Filters used to minimize intersymbol interference in digital communication systems have gain characteristics that approximate a Gaussian or a ”raised-cosine” curve. Then there are situations in which we wish to insert a 2-port between a signal source and a fixed impedance load in order to maximize power transfer to the load, which may be a function of frequency, as in matching a power amplifier to a transmitting antenna. In addition, circuits are needed to equalize the group delay of a bandpass filter so that the overall delay will be constant over the passband in order to reduce delay distortion of the signal. In all these situations, there is no analytic way to design such circuits. One can always resort to trial and error, but not only this will be inefficient, but also we will never know how good the final design is.

Fortunately, we can formulate the design problem as an optimization problem and a rich repertoire of methods exist that can help us find a design to meet just about any practical specifications. In this chapter we will study a few basic methods that have been found to be useful for circuit design and that are easy to implement.

11.1 Formulation of design problem

Let $\mathbf{x}$ be a vector of n design parameters such as element values, time constants, critical frequencies, poles or zeros of a circuit function. For a given $\mathbf{x}$, let $F(\mathbf{x},\omega)$ be a real function of $\mathbf{x}$ and ω. $F(\mathbf{x},\omega)$ could be some circuit response, such as the gain, phase, or group delay of a transfer function, or the real or imaginary part of an impedance function. We assume that given $\mathbf{x}$ and ω, $F(\mathbf{x},\omega)$ can be computed. It is not necessary that the analytic or symbolic form of $F(\mathbf{x},\omega)$ be known. Let $F_D(\omega)$ be a desired response. Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ be a set of discrete frequency points. We define an objective function $\phi(\mathbf{x})$ as

$$\phi(\mathbf{x}) = \sum_{\omega_k \in \Omega} W(\omega_k)\left[F(\mathbf{x},\omega_k) - F_D(\omega_k)\right]^2 , \tag{11.1}$$

where $W(\omega)$ is some weighting function to emphasize or de-emphasize the circuit response over certain frequency range. It is seen in this case, the objective function is a sum of squares of the differences of the actual response and desired response. The design problem is to find $\mathbf{x}^*$ such that

$$\phi(\mathbf{x}^*) = \min_{\mathbf{x}} \phi(\mathbf{x}), \tag{11.2}$$

namely, $\mathbf{x}^*$ is the vector of design parameters which minimizes the objective function $\phi(\mathbf{x})$ over all possible values of $\mathbf{x}$. If $\mathbf{x}^*$ exists, we declare it as a solution. In general $\phi(\mathbf{x})$ is a complicated function of $\mathbf{x}$, and we may end up at a local minimum.

If the design parameters are element values, e_k, $k = 1,\ldots,n$, which must be positive, we may define $x_k = \ln e_k$ to ensure that whatever $\mathbf{x}^*$ is, e_k will be positive.

Other forms of the objective function are possible. For example, if we define

$$\phi(\mathbf{x}) = \max_{\omega_k \in \Omega} \left[W(\omega_k)\left|F(\mathbf{x},\omega_k) - F_D(\omega_k)\right|\right] , \tag{11.3}$$

and define the solution $\mathbf{x}^*$ such that

$$\phi(\mathbf{x}^*) = \min_{\mathbf{x}} \phi(\mathbf{x}), \tag{11.4}$$

we are trying to minimize the maximum difference of the actual response and desired response. It is often called a minimax problem, which in practice can be formulated as: Find $\mathbf{x}^*$ such that

$$\begin{aligned} \phi(\mathbf{x}^*) &= \min_{\mathbf{x}} \phi(\mathbf{x}) \\ \phi(\mathbf{x}) &= \left[\sum_{\omega_k \in \Omega} W(\omega_k)\left|F(\mathbf{x},\omega_k) - F_D(\omega_k)\right|^p\right]^{1/p} \end{aligned} \tag{11.5}$$

for some large integer p.

11.2 Solution strategy

The objective function $\phi(\mathbf{x})$ defines a surface in an n-dimensional (parameter) space. Values of $\mathbf{x}$ such that $\phi(\mathbf{x}) =$ constant define a contour on the surface. Our object is to find a point in this space where $\phi(\mathbf{x})$ is minimum. There are scores of algorithms that will take us there. The underlying technique of all of these algorithms is that we start at an initial point $\mathbf{x}^0$ and generate a sequence of points $\mathbf{x}^1, \mathbf{x}^2, \ldots$ such that $\phi(\mathbf{x}^{k+1}) < \phi(\mathbf{x}^k)$. The rationale is that as long as we go downhill at every step, we should arrive at a minimum eventually. One algorithm differs from another in the way the direction and step size are determined and in the way we decide if we have reached a minimum.

In this chapter, we will study the following methods:

1. Steepest descent,
2. Newton, and
3. Least squares.

11.3 Steepest descent

If we want to go downhill, we should know the "lay of the land" around us. Suppose we have generated a sequence $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$. In the neighborhood of $\mathbf{x}^k$, let the objective function $\phi(\mathbf{x})$ be represented by a three-term Taylor series, namely, a quadratic:

$$\phi(\mathbf{x}) = \phi(\mathbf{x}^k) + (\nabla\phi(\mathbf{x}^k))^T(\mathbf{x} - \mathbf{x}^k) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^k)^T H(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k), \tag{11.6}$$

where $\nabla\phi(\mathbf{x})$ is the gradient of $\phi(\mathbf{x})$:

$$\nabla\phi(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial\phi}{\partial x_1} \\ \dfrac{\partial\phi}{\partial x_2} \\ \vdots \\ \dfrac{\partial\phi}{\partial x_n} \end{bmatrix}, \tag{11.7}$$

and $H(\mathbf{x})$ is the Hessian matrix of $\phi(\mathbf{x})$:

$$H(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2\phi}{\partial x_1 \partial x_n} \\ \frac{\partial^2\phi}{\partial x_2 \partial x_1} & \frac{\partial^2\phi}{\partial x_2^2} & \cdots & \frac{\partial^2\phi}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2\phi}{\partial x_n \partial x_1} & \frac{\partial^2\phi}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2\phi}{\partial x_n^2} \end{bmatrix}. \tag{11.8}$$

Let the next iterate be $\mathbf{x}^{k+1}$:

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \Delta\mathbf{x}^k = \mathbf{x}^k + t^k\mathbf{u}^k\,, \tag{11.9}$$

where $\Delta\mathbf{x}^k$ is the step that takes us from $\mathbf{x}^k$ to $\mathbf{x}^{k+1}$, $\mathbf{u}^k$ is a unit vector in the direction of $\Delta\mathbf{x}^k$ and t^k is the length of the step, a positive scalar. For small t^k, we can write

$$\phi(\mathbf{x}^{k+1}) = \phi(\mathbf{x}^k) + (\nabla\phi(\mathbf{x}^k))^T \cdot \mathbf{u}^k t^k\,. \tag{11.10}$$

If we wish to go downhill

$$\phi(\mathbf{x}^{k+1}) < \phi(\mathbf{x}^k)\,, \tag{11.11}$$

which requires that

$$(\nabla\phi(\mathbf{x}^k))^T \cdot \mathbf{u}^k < 0\,, \tag{11.12}$$

namely, the next step should be in a direction opposite to that of the gradient:

$$\mathbf{u}^k = -\frac{\nabla\phi(\mathbf{x}^k)}{\parallel \nabla\phi(\mathbf{x}^k) \parallel}\,. \tag{11.13}$$

If the surface at $\mathbf{x}^k$ can indeed be represented by a quadratic as in Eq. (11.6), and if we follow the direction $\mathbf{u}^k$ downhill, we can find the minimum along this direction in one step. Let the step size be t^k. Along $\mathbf{u}^k$, $\phi(\mathbf{x})$ is a function of t^k and we can write

$$\phi(t^k) = \phi(\mathbf{x}^k + t^k\mathbf{u}^k) = \phi(\mathbf{x}^k) + t^k(\nabla\phi(\mathbf{x}^k))^T \cdot \mathbf{u}^k + \frac{1}{2}(t^k)^2(\mathbf{u}^k)^T H(\mathbf{x}^k)\mathbf{u}^k\,. \tag{11.14}$$

At the minimum,

$$\frac{\partial\phi(t^k)}{\partial t^k} = 0\,, \tag{11.15}$$

and the step size is given by

$$t^k = \frac{(\nabla\phi(\mathbf{x}^k))^T\nabla\phi(\mathbf{x}^k)}{(\nabla\phi(\mathbf{x}^k))^T H(\mathbf{x}^k)\nabla\phi(\mathbf{x}^k)} \parallel \nabla\phi(\mathbf{x}^k) \parallel\,. \tag{11.16}$$

We set the next iterate to be

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \frac{(\nabla\phi(\mathbf{x}^k))^T\nabla\phi(\mathbf{x}^k)}{(\nabla\phi(\mathbf{x}^k))^T H(\mathbf{x}^k)\nabla\phi(\mathbf{x}^k)}\nabla\phi(\mathbf{x}^k)\,, \tag{11.17}$$

and continue the iteration from there. The step will be in the negative gradient direction if

$$(\nabla\phi(\mathbf{x}^k))^T H(\mathbf{x}^k)\nabla\phi(\mathbf{x}^k) > 0\,, \tag{11.18}$$

namely, if the Hessian matrix is positive definite.

In general, the surface is not quadratic and if we take the step to $\mathbf{x}^{k+1}$ according to Eq. (11.17), we may not be at a minimum. We take the step anyway because we are at a point where the objective function has a lower value. We repeat the process from there of computing the gradient, Hessian, unit directional vector and step size, until a minimum of the objective function is reached, or until some stopping criterion is met, for example,

$$\left|\phi(\mathbf{x}^{k+1}) - \phi(\mathbf{x}^k)\right| < \varepsilon \tag{11.19}$$

over a long sequence of $\phi(\mathbf{x}^k)$.

In practice, it often happens that for some k,

$$\phi(\mathbf{x}^{k+1}) > \phi(\mathbf{x}^k)\,, \tag{11.20}$$

indicating that the Hessian matrix may not be positive definite at $\mathbf{x}^k$, or the surface is poorly represented by a quadratic. In this case, we discard $\mathbf{x}^{k+1}$ and take small steps from $\mathbf{x}^k$. In other words, we will "hug" the surface and try to go downhill "gingerly."

The strategy is to do a one-dimensional search for a minimum of the objective function along the negative gradient direction not too far from $\mathbf{x}^k$. Then take the next iterate to be where the minimum occurs.

11.3.1 One-dimensional search

We start at $\mathbf{x}^k$ and take a few small steps along $\mathbf{u}^k$. Evaluate the objective function at these steps and pass a polynomial through the values of the objective function. Find the minimum of the polynomial and set the value of $\mathbf{x}$ at the minimum to be $\mathbf{x}^{k+1}$.

We will use a quadratic polynomial. Let

$$\phi(\lambda) = \phi(\mathbf{x}^k + \lambda\mathbf{u}^k)\,. \tag{11.21}$$

Generate a sequence $\{\lambda^1 < \lambda^2 < \ldots\}$ with the following algorithm:

$$\lambda^0 = 0\,, \tag{11.22}$$

$$\lambda^i = \sum_{j=1}^{i} 2^{j-1}\cdot\delta \quad \text{for } i = 1, 2, \ldots \tag{11.23}$$

where δ is a small fraction of t^k given in Eq. (11.16). Correspondingly, compute the sequence $\{\phi(\lambda^1) > \phi(\lambda^2) > \ldots\}$. If it occurs that for some i

$$\phi(\lambda^{i-1}) < \phi(\lambda^i)\,, \tag{11.24}$$

we stop this sequence and compute

$$\lambda^{i+1} = \lambda^{i-1} + (\lambda^{i-1} - \lambda^{i-2}), \tag{11.25}$$

and evaluate $\phi(\lambda^{i+1})$. Note that the four points λ^{i-2}, λ^{i-1}, λ^{i+1}, and λ^{i} are equally-spaced. There are two situations to consider:

1. $\phi(\lambda^{i+1}) < \phi(\lambda^{i-1})$: There exists a minimum in the interval $(\lambda^{i-1}, \lambda^{i})$. Let $\lambda_a = \lambda^{i-1}$, $\lambda_b = \lambda^{i+1}$, $\lambda_c = \lambda^{i}$.
2. $\phi(\lambda^{i+1}) > \phi(\lambda^{i-1})$: There exists a minimum in the interval $(\lambda^{i-2}, \lambda^{i+1})$. Let $\lambda_a = \lambda^{i-2}$, $\lambda_b = \lambda^{i-1}$, $\lambda_c = \lambda^{i+1}$.

Pass a quadratic through $\phi(\lambda_a)$, $\phi(\lambda_b)$, and $\phi(\lambda_c)$. The minimum occurs at

$$\lambda_{min} = \lambda_b + \frac{(\lambda_c - \lambda_b)(\phi(\lambda_a) - \phi(\lambda_c))}{2(\phi(\lambda_a) - 2\phi(\lambda_b) + \phi(\lambda_c))}. \tag{11.26}$$

We then set

$$\mathbf{x}^{k+1} = \mathbf{x}^{k} + \lambda_{min}\mathbf{u}^{k}, \tag{11.27}$$

and continue the steepest descent from there.

11.4 Newton's method

The steepest descent method works well when we are far away from a minimum. Often the descent is rapid especially at the beginning of the minimization process. When we are near a minimum where the gradient is small, this method often produces very small steps so that the reduction of the objective function is negligible and we seem to be expending efforts for no gain.

When such a situation arises, we should switch to Newton's method which works well near a minimum. At a minimum, the gradient of the objective function is zero. The minimization problem becomes one of finding $\mathbf{x}^*$ such that

$$\nabla\phi(\mathbf{x}^*) = 0. \tag{11.28}$$

In general $\nabla\phi(\mathbf{x})$ is a nonlinear function of $\mathbf{x}$ and the problem is one of solving a set of n nonlinear algebraic equations in n unknowns. The best known method for this is Newton iteration.

Starting with an initial guess $\mathbf{x}^0$, we generate a sequence $\{\mathbf{x}^1, \mathbf{x}^2, \ldots\}$ according to the following algorithm

$$\mathbf{x}^{k+1} = \mathbf{x}^{k} + \Delta\mathbf{x}, \tag{11.29}$$

where $\Delta\mathbf{x}$ is solved from

$$H(\mathbf{x}^k)\,\Delta\mathbf{x} = -\nabla\phi(\mathbf{x}^k), \tag{11.30}$$

where $H(\mathbf{x})$ is the Jacobian of $\nabla\phi(\mathbf{x})$, or the Hessian matrix of $\phi(\mathbf{x})$.

When the length of $\Delta\mathbf{x}$ is negligibly small, we stop and declare we have convergence. Newton's method converges quadratically in the neighborhood of a zero. This is why it is effective when we are near a minimum of $\phi(\mathbf{x})$.

When convergence is slow, we should again take smaller steps and the algorithm is modified to be

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \lambda^k \Delta\mathbf{x}, \tag{11.31}$$

where λ^k is a small positive parameter which may change from step to step.

11.5 Least squares method

Let us return to the minimization problem and look at it in a slightly different way. With reference to Eq. (11.1), without loss of generality, let the weighting function be absorbed in $F(\mathbf{x},\omega)$ and $F_D(\omega)$. We re-state the problem as: Find $\mathbf{x}^*$ such that

$$\phi(\mathbf{x}^*) = \min_{\mathbf{x}} \phi(\mathbf{x}), \tag{11.32}$$

where

$$\phi(\mathbf{x}) = \sum_{i=1}^{m} [F(\mathbf{x},\omega_i) - F_D(\omega_i)]^2 . \tag{11.33}$$

This is known as the nonlinear least squares problem. Define a vector function $F(\mathbf{x})$:

$$F(\mathbf{x}) = \begin{bmatrix} F(\mathbf{x},\omega_1) - F_D(\omega_1) \\ F(\mathbf{x},\omega_2) - F_D(\omega_2) \\ \cdots \\ F(\mathbf{x},\omega_m) - F_D(\omega_m) \end{bmatrix} = \begin{bmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_m(\mathbf{x}) \end{bmatrix} . \tag{11.34}$$

The objective function can be written as

$$\phi(\mathbf{x}) = [F(\mathbf{x})]^T F(\mathbf{x}), \tag{11.35}$$

and its gradient is

$$\nabla\phi(\mathbf{x}) = 2[J(\mathbf{x})]^T \times F(\mathbf{x}), \tag{11.36}$$

where $J(\mathbf{x})$ is the $m \times n$ Jacobian of $F(\mathbf{x})$:

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix} . \tag{11.37}$$

Starting with an initial point $\mathbf{x}^0$, we generate a sequence $\mathbf{x}^1, \mathbf{x}^2, \ldots$. The iterate $\mathbf{x}^{k+1}$ is computed from

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \Delta\mathbf{x}^k, \tag{11.38}$$

where $\Delta\mathbf{x}^k$ satisfies

$$F(\mathbf{x}^{k+1}) = F(\mathbf{x}^k) + J(\mathbf{x}^k)\Delta\mathbf{x}^k, \tag{11.39}$$

which is a two-term Taylor series approximation of $F(\mathbf{x}^{k+1})$. Assume the objective function $\phi(\mathbf{x})$ is smooth. Then its gradient at $\mathbf{x}^{k+1}$ can be approximated as:

$$\nabla\phi(\mathbf{x}^{k+1}) = 2[J(\mathbf{x}^k)]^T \times F(\mathbf{x}^{k+1}). \tag{11.40}$$

Combining the last two equations, we get an updating equation:

$$[J(\mathbf{x}^k)]^T [J(\mathbf{x}^k)]\Delta\mathbf{x}^k = -[J(\mathbf{x}^k)]^T F(\mathbf{x}^k). \tag{11.41}$$

Solving the system of linear equations to get $\Delta\mathbf{x}^k$ and substituting into Eq. (11.39), we obtain $\mathbf{x}^{k+1}$. The iteration continues until we reach convergence.

It is possible that the system of equations Eq. (11.41) is ill-conditioned. The length of the step $\Delta\mathbf{x}$ then becomes too large and we may not be going downhill, i.e., for some k:

$$\phi(\mathbf{x}^{k+1}) > \phi(\mathbf{x}^k). \tag{11.42}$$

Levenberg and Marquardt [43] proposed that $\Delta\mathbf{x}^k$ be found from a modified updating equation:

$$\left([J(\mathbf{x}^k)]^T [J(\mathbf{x}^k)] + \lambda^k I\right)\Delta\mathbf{x}^k = -[J(\mathbf{x}^k)]^T F(\mathbf{x}^k), \tag{11.43}$$

where I is an $n \times n$ identity matrix and λ^k is a positive scalar whose value is adjusted from iteration to iteration to prevent the occurrence of (11.42).

11.6 Remarks

We have presented three methods to find a minimum of an objective function. It is possible that the minimum we found is a local rather than a global minimum. There is really not any good way to determine if we have a local or global minimum. If the "optimum" set of design parameters are found to yield a response sufficiently close to the desired response, we can declare we have found the solution to the design problem.

The steepest descent and Newton's methods both require that we compute the Hessian matrix of the objective function. The terms of the matrix are second derivatives with respect to each and every parameter. The computational cost is substantial. Moreover, for these methods to be effective, the Hessian matrix must be positive definite. To test if a matrix is positive definite requires another set of computation. It seems that it would be desirable if we could find a method which does not require the Hessian matrix. For circuit design, the Levenberg-Marquardt method is popular, even for nonlinear circuit design. It does require that we have the Jacobian, but as we see shortly, the first derivative terms can be computed from solutions of a set of node equations and its transpose.

It must be added that there are other minimization methods. The reader is urged to consult standard texts on optimization techniques. One method worthy of mention is the Davidson-Fletcher-Powell (DFP) algorithm [51] which at each iteration generates an approximation of the Hessian (actually its inverse) that is guaranteed to be positive definite.

11.7 Computation of gradient and Hessian matrix

All minimization methods require that we compute the gradient of an objective function. In the steepest descent and Newton's methods, we need the Hessian matrix as well. For linear circuits, the computation is straightforward and numerical differentiation is not necessary.

Consider a circuit of $N+1$ nodes with the ground node labeled 0. Let a voltage source of 1 V be connected to node 1 and ground and take the output voltage at node N. We describe the circuit by its modified node equations:

$$\begin{bmatrix} Y & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V \\ I_E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{or} \tag{11.44}$$

$$Y_m u = e_{N+1}\,, \tag{11.45}$$

where Y is the $N \times N$ node admittance matrix, V the vector of node voltages, I_E is the current in the voltage source, and e_{N+1} is an $N+1$ vector of zeros except that its last element is one. Let the vector of design parameters be

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^T\,, \tag{11.46}$$

where $x_k = \ln C_k$ represents capacitance C_k, and similarly for inductances and conductances.

We will consider optimization of the circuit response in terms of the gain or phase of the output voltage $V_N(\mathbf{x}, j\omega)$. Let

$$V_N(\mathbf{x}, j\omega) = e^{\alpha(\mathbf{x},\omega)+j\beta(\mathbf{x},\omega)}\,. \tag{11.47}$$

We define the gain objective function to be

$$\phi(\mathbf{x}) = \sum_{\omega_k \in \Omega} \left\{ W_\alpha(\omega_k)\,[\alpha(\mathbf{x}, \omega_k) - \alpha_D(\omega_k)]^2 \right\}. \tag{11.48}$$

Its gradient with respect to a parameter x_s is

$$\frac{\partial \phi}{\partial x_s} = 2 \sum_{\omega_k \in \Omega} \left\{ W_\alpha(\omega_k) \left[\alpha(\mathbf{x}, \omega_k) - \alpha_D(\omega) \right] \frac{\partial \alpha}{\partial x_s} \right\}, \tag{11.49}$$

and the (r,s) component of the Hessian matrix is

$$\frac{\partial^2 \phi}{\partial x_r \partial x_s} = 2 \sum_{\omega_k \in \Omega} \left\{ W_\alpha(\omega_k) \left[\left(\frac{\partial \alpha}{\partial x_r} \right) \left(\frac{\partial \alpha}{\partial x_s} \right) + \left[\alpha(\mathbf{x}, \omega_k) - \alpha_D(\omega_k) \right] \frac{\partial^2 \alpha}{\partial x_r \partial x_s} \right] \right\}. \tag{11.50}$$

Similarly, the phase objective function is

$$\phi(\mathbf{x}) = \sum_{\omega_k \in \Omega} \left\{ W_\beta(\omega_k) \left[\beta(\mathbf{x}, \omega_k) - \beta_D(\omega_k) \right]^2 \right\}. \tag{11.51}$$

Its gradient with respect to a parameter x_s is

$$\frac{\partial \phi}{\partial x_s} = 2 \sum_{\omega_k \in \Omega} \left\{ W_\beta(\omega_k) \left[\beta(\mathbf{x}, \omega_k) - \beta_D(\omega) \right] \frac{\partial \beta}{\partial x_s} \right\}, \tag{11.52}$$

and the (r,s) component of the Hessian matrix is

$$\frac{\partial^2 \phi}{\partial x_r \partial x_s} = 2 \sum_{\omega_k \in \Omega} \left\{ W_\beta(\omega_k) \left[\left(\frac{\partial \beta}{\partial x_r} \right) \left(\frac{\partial \beta}{\partial x_s} \right) + \left[\beta(\mathbf{x}, \omega_k) - \beta_D(\omega_k) \right] \frac{\partial^2 \beta}{\partial x_r \partial x_s} \right] \right\}. \tag{11.53}$$

Note that it is possible to define an objective function that combines gain and phase together as a weighted sum of the two objective functions. But as we learned in Chapter Four, the gain and phase of a minimum-phase transfer function are not independent, and we should only proceed with such a minimization problem with caution.

The partial derivatives in the gradient and Hessian matrix are given by

$$\frac{\partial \alpha}{\partial x_s} = \Re \left[\frac{1}{V_N} \frac{\partial V_N}{\partial x_s} \right]. \tag{11.54}$$

$$\frac{\partial \beta}{\partial x_s} = \Im \left[\frac{1}{V_N} \frac{\partial V_N}{\partial x_s} \right]. \tag{11.55}$$

$$\frac{\partial^2 \alpha}{\partial x_r \partial x_s} = \Re \left[\frac{1}{V_N} \frac{\partial^2 V_N}{\partial x_r \partial x_s} - \frac{1}{V_N^2} \left(\frac{\partial V_N}{\partial x_r} \right) \left(\frac{\partial V_N}{\partial x_s} \right) \right]. \tag{11.56}$$

$$\frac{\partial^2 \beta}{\partial x_r \partial x_s} = \Im \left[\frac{1}{V_N} \frac{\partial^2 V_N}{\partial x_r \partial x_s} - \frac{1}{V_N^2} \left(\frac{\partial V_N}{\partial x_r} \right) \left(\frac{\partial V_N}{\partial x_s} \right) \right]. \tag{11.57}$$

It remains to find the first and second order sensitivity functions $\partial V_n / \partial x_s$, $\partial V_n / \partial x_r$ and $\partial^2 V_N / \partial x_r \partial x_s$.

11.7.1 Sensitivity functions

We have already derived the sensitivity functions in Chapter Four in connection with computation of group delay. We will re-do the derivation briefly and then proceed to derive the second order sensitivities.

Start with the modified node equations Eq. (11.45):

$$Y_m u = e_{N+1} \,. \tag{11.58}$$

Let y_s be an admittance connected between node i_s and node j_s. Let V_N be the output voltage. Then as shown in Sect. 4.9.1, the change in V_N with respect to a small change in y_s is given by:

$$\frac{dV_N}{dy_s} = -(q_{i_s} - q_{j_s})(u_{i_s} - u_{j_s}) \,, \tag{11.59}$$

where u_k is the voltage at node k from the solution of Eq. (11.58), and q_k is the kth element of the solution vector q in:

$$Y_m^T q = e_N \,, \tag{11.60}$$

which is a system of modified node equations of the adjoint circuit.

Next, we consider the second order sensitivity. Let y_r be an admittance connected between nodes i_r and j_r. Differentiation of Eq. (11.59) with respect to y_r yields:

$$\frac{d^2 V_N}{dy_r\, dy_s} = -(u_{i_s} - u_{j_s}) \frac{d}{dy_r}(q_{i_s} - q_{j_s}) - (q_{i_s} - q_{j_s}) \frac{d}{dy_r}(u_{i_s} - u_{j_s}) \,. \tag{11.61}$$

Since u_k and q_k are node voltages, the derivative terms can be found in the same way as dV_N/dy_s, and we obtain:

$$\frac{d^2 V_N}{dy_r\, dy_s} = (p_{i_r} - p_{j_r})(q_{i_r} - q_{j_r})(u_{i_s} - u_{j_s}) + (q_{i_s} - q_{j_s})(t_{i_r} - t_{j_r})(u_{i_r} - u_{j_r}) \,, \tag{11.62}$$

where p_k is the kth element of the vector p found from:

$$Y_m p = e_{i_s, j_s} \,, \tag{11.63}$$

and t_k is the kth element of the vector t found from:

$$Y_m^T t = e_{i_s, j_s} \,, \tag{11.64}$$

where $e_{k,\ell}$ is a vector of zeros except that element k is 1 and element ℓ is -1. In words, $p_{i_r} - p_{j_r}$ is the voltage across admittance y_r when a current source of 1A is connected across admittance y_s, and $t_{i_r} - t_{j_r}$ is the voltage across y_r of the adjoint circuit when a current source of 1A is connected across y_s. We do not actually have to construct the adjoint circuit. Solving Eq. (11.64) for t will do.

It is seen that the terms of the gradient and Hessian matrix can be found by solving a series of node equations, without taking derivatives.

Lastly, if $y_s = j\omega C_s$ and $x_s = \ln C_s$, then from Eq. (11.54) we get:

$$\frac{\partial \alpha}{\partial x_s} = \Re\left[\frac{-y_s}{V_N(\mathbf{x},\omega)}(q_{i_s} - q_{j_s})(u_{i_s} - u_{j_s})\right]_{y_s = j\omega C_s} . \tag{11.65}$$

If $y_s = 1/(j\omega L_s)$ and $x_s = \ln L_s$, then

$$\frac{\partial \alpha}{\partial x_s} = \Re\left[\frac{y_s}{V_N(\mathbf{x},\omega)}(q_{i_s} - q_{j_s})(u_{i_s} - u_{j_s})\right]_{y_s = 1/(j\omega L_s)} . \tag{11.66}$$

and similarly for phase sensitivities (see Eq. (11.55)).

11.8 Examples of design by optimization

Optimization software packages exist and are widely available.[1] For simple circuit design and to see how the convergence process progresses iteration by iteration, it is best to write one's own program. In what follows, we present two examples, one on the design of a non-conventional filter and one on broadband matching.

Example 11.1. : Gaussian low-pass filter.

Let us find a low-pass filter whose frequency response approximates that of a Gaussian response. Let the gain of the filter be:

$$\alpha(\omega) = \ln\left|\frac{V_N(j\omega)}{E}\right| , \tag{11.67}$$

and let the desired response be:

$$\alpha_D(\omega) = \ln\left[\frac{1}{2}e^{-\omega^2/2}\right] . \tag{11.68}$$

We choose a fifth order LC ladder terminated in 1Ω at both ends, as shown in Fig. 11.1. The design parameters are the five element values: $e = [C_1, L_2, C_3, L_4, C_5]$.

We will use the Levenberg-Marquardt minimization method. Setting up the objective function as in Eq. (11.33), we have:

$$\phi(\mathbf{x}) = [F(\mathbf{x})]^T F(\mathbf{x}) . \tag{11.69}$$

[1] In MATLAB®, they are found in Optimization Toolbox.

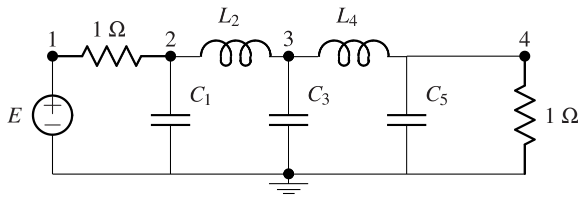


Fig. 11.1 A fifth order low-pass filter to approximate a Gaussian response

$$F(\mathbf{x}) = \begin{bmatrix} F(\mathbf{x},\omega_1) - F_D(\omega_1) \\ F(\mathbf{x},\omega_2) - F_D(\omega_2) \\ \cdots \\ F(\mathbf{x},\omega_m) - F_D(\omega_m) \end{bmatrix} = \begin{bmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_m(\mathbf{x}) \end{bmatrix}, \tag{11.70}$$

where

$$F_k(\mathbf{x}) = \ln|V_4(\mathbf{x},\omega_k)| + \ln 2 + \omega_k^2/2. \tag{11.71}$$

Starting with an initial guess: $e = [0.2, 1.0, 1.0, 2.0, 0.2]$ and taking 27 points in the frequency interval: $[0.1, 2]$, after 100 iterations, we obtain a convergence history as shown in Fig. 11.2. We see the error decreases monotonically, initially rapidly and then slowly after 60 iterations. At each iteration, the step size actually taken is half

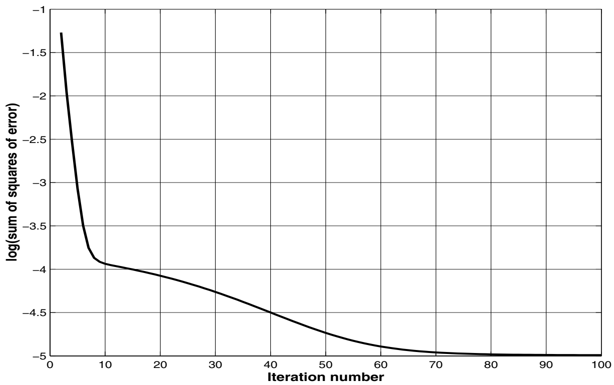


Fig. 11.2 Convergence history of minimization by the Levenberg-Marquardt method

of what is computed in Eq. (11.41). In addition, the coefficient λ_k in Eq. (11.43) is set to be the maximum of the matrix elements in absolute value of $[J(\mathbf{x})]^T J(\mathbf{x})$ if at some iteration the objective function increases, and to 0.2 if it does not. The final values of the elements are:

$$C_1 = 0.3376\,\mathrm{F}, \qquad L_2 = 0.7772\,\mathrm{H}, \qquad C_3 = 1.1787\,\mathrm{F}$$
$$L_4 = 2.8778\,\mathrm{H}, \qquad C_5 = 0.2204\,\mathrm{F}.$$

Figure 11.3 shows the frequency response of the filter compared to the Gaussian response. It is seen that the two responses match well up to about $\omega = 3$. The two

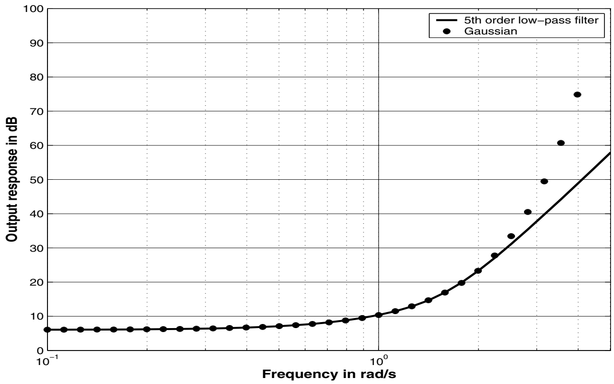


Fig. 11.3 Frequency response of filter compared to the desired Gaussian response

begin to deviate significantly at $\omega = 5$. This is to be expected. The transfer function of an *RLC* circuit is a rational function. At high frequencies, its magnitude decreases as some negative power of ω. On the other hand, the Gaussian response decreases exponentially. Therefore the filter response will not approximate the Gaussian response well at high frequencies. However, the output signal is already highly attenuated there and it does not matter if the filter response does not match the Gaussian response where the signal is small.

Example 11.2. : Broadband matching.

So far, in all our filter designs, the load impedance is a resistor. In practice, the most common type of load impedance is a resistor in parallel with a capacitor, which represents the unavoidable parasitic capacitance at the load. In many applications, we wish to transfer as much signal power from the source to the load as possible over some passband. Our problem then is to find a lossless two-port inserted between the source impedance, which is usually a resistor, and the load such that the overall frequency response is that of some desirable characteristic such as maximally flat or equal-ripple.

Consider designing an *LC* ladder placed between a source with an internal resistance of 1Ω and a load impedance consisting of a 5Ω resistor in parallel with a $1\mathrm{F}$ capacitor. It is required that the overall response be that of a fifth order Butterworth. The circuit is shown in Fig. 11.4 and our problem is to find a set of element values $e = [C_1, L_2, C_3, L_4, C_5]$ such that the gain of the circuit:

$$\alpha(\omega) = \ln \left| \frac{V_4(j\omega)}{E} \right| , \tag{11.72}$$

approximates the Butterworth response:

$$\alpha_D(\omega) = 0.5 \ln \left[\frac{25/36}{1+\omega^{10}} \right] . \tag{11.73}$$

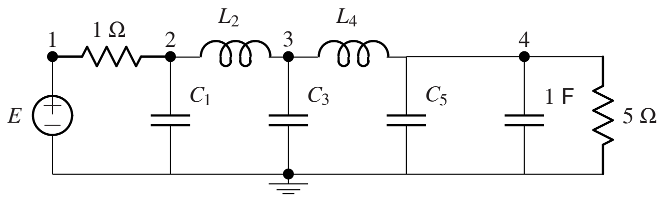


Fig. 11.4 A fifth order low pass filter to match a complex load and to have a Butterworth response

We follow the same procedure as in Example 11.1. The only change is to replace the Gaussian response with a Butterworth response. The convergence history and frequency response are shown in Fig. 11.5. After 60 iterations, the error is less than 1×10^{-6}. The frequency response of the low-pass filter matches the Butterworth response well even at high frequencies.

The optimized element values are:

$$C_1 = 0.3204\,\mathsf{F}, \qquad L_2 = 0.9302\,\mathsf{H}, \qquad C_3 = 1.4366\,\mathsf{F},$$
$$L_4 = 1.7590\,\mathsf{H}, \qquad C_5 = 0.5868\,\mathsf{F}.$$

11.9 Remarks

We have presented techniques for circuit design by optimization. These techniques are applicable with little modification to any linear circuits, in particular, small-signal equivalent electronic circuits. The only difference is that the modified node equations of these circuits are not symmetric, as noted in Sect. 4.3.

Circuit design by optimization is often used to construct a small-signal circuit model of an electronic device, for example, the transistor. The internal structure of a transistor cannot be realistically represented by a circuit, but its frequency response at the output can be measured. We may propose a two-port model and determine its element values by optimization to match the measured data. The optimized two-port model is taken to be a good representation of the transistor over the measured frequency range insofar as its terminal behaviors are concerned, and the model is used as an equivalent circuit of the transistor for analysis and design.

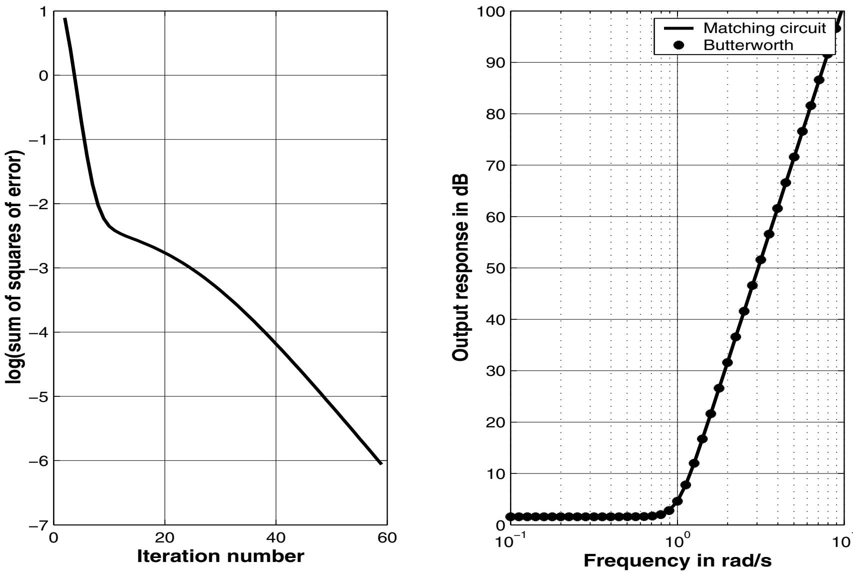


Fig. 11.5 Convergence history and frequency response of broadband matching circuit designed by optimization

Lastly, as we see in the last example, optimization techniques can be used to design circuits for broadband matching. There are mathematical theories which guide us in the synthesis of matching circuits [23, 24, 70, 17, 72], but design by optimization is simpler and more flexible.

Problems

11.1. Derive the expression for second order sensitivity as given in Eq. (11.62).

11.2. With reference to Example 11.1, re-do the design with a different initial guess of the element values. It is likely that the optimized element values are different from what were obtained. Explain.

11.3. *Gaussian low-pass filter*: With reference to Example 11.1, re-do the design with a seventh order low-pass filter. What is the frequency range over which the error of approximation is less that 10^{-4}?

11.4. *Raised cosine low-pass filter*[39]: Design a low-pass filter to have a frequency response that approximates the following transmission power gain function:

$$|S_{21}(j\omega)|^2 = \begin{cases} 1 & \text{for } 0 \le \omega \le \omega_o; \\ \frac{1}{2}\left[1+\cos\left(\frac{\pi}{2}\frac{\omega-\omega_o}{\omega_o}\right)\right] & \text{for } \omega_o \le \omega \le 3\omega_o; \\ 0 & \text{for } \omega > 3\omega_o. \end{cases} \tag{11.74}$$

Experiment with a fifth and seventh order *LC* ladder terminated in 1Ω at both ends. Can you explain why it is so difficult to find a good solution? Such a filter is used in a digital communication system to shape a pulse so as to minimize intersymbol interference.

11.5. *Broadband matching*: Find a lossless two-port inserted between a resistor of 1Ω and a load impedance which consists of an inductor of 0.5H in series with a parallel combination of a 5Ω resistor and a 0.1F capacitor. The overall gain response across the load impedance is to be a fifth order Butterworth response.

11.6. *Broadband matching*: A power amplifier with internal resistance of 5 Ω is to drive a load consisting of a resistor of 50 Ω in parallel with a capacitor of 1 pF. Insert a series inductor of inductance L and a shunt capacitor of capacitance C between the amplifier and load. Find optimum values for L and C such that the overall transmission power gain approximates that of a third order Butterworth filter.

11.7. *Order reduction*: Order reduction is a technique by which an Nth order circuit is approximated or modeled by a circuit of order less than N, in the sense that their frequency or step responses match in some way. Consider a five-section *RC* line. The line is essentially a fifth order low-pass filter. Let us model it with a third order *RLC* circuit, shown in Fig. 11.6 together with the *RC* line. Use the Levenberg-Marquardt optimization method to find values of the elements of the third order circuit such that its frequency response approximates that of the fifth *RC* line in the least squares sense over the frequency range $0.01 \le \omega \le 4$. Let $R = 1\,\Omega$ and $C = 1$ F. With starting values of the elements at $C_1 = 1$ F, $L = 2$ H, and $C_2 = 1$ F, we find an optimized set of values given by $C_1 = 0.7659$ F, $L = 1.9747$ H, and $C_2 = 14.3684$ F, after 50 iterations. For this set of values, the frequency response and step response of the third order *RLC* are shown in Fig. 11.7, together with those of the fifth order *RC* line. We see that the two corresponding responses match each other quite well. Now repeat the problem with a ten-section *RC* line and model it with a fifth order *RLC* circuit. If successful, it means that a tenth order circuit can be represented by a fifth order circuit insofar as its frequency and step responses are concerned. For applications in digital circuits, such reduction of order of representation is much desired to simplify analysis. This technique can be extended to a general *RC* circuit of any complexity. Conversely, we say that a fifth order *RLC* circuit with two inductors can be replaced by a tenth order *RC* circuit with no inductors, and it has essentially the same frequency and step responses as the *RLC* circuit.

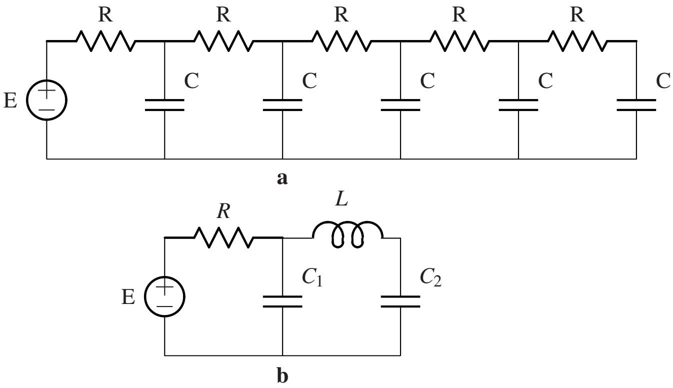


Fig. 11.6 A fifth order *RC* line (**a**) is to be modeled by a third order *RLC* circuit (**b**) such that their frequency responses are matched in the least squares sense (Problem 11.7)

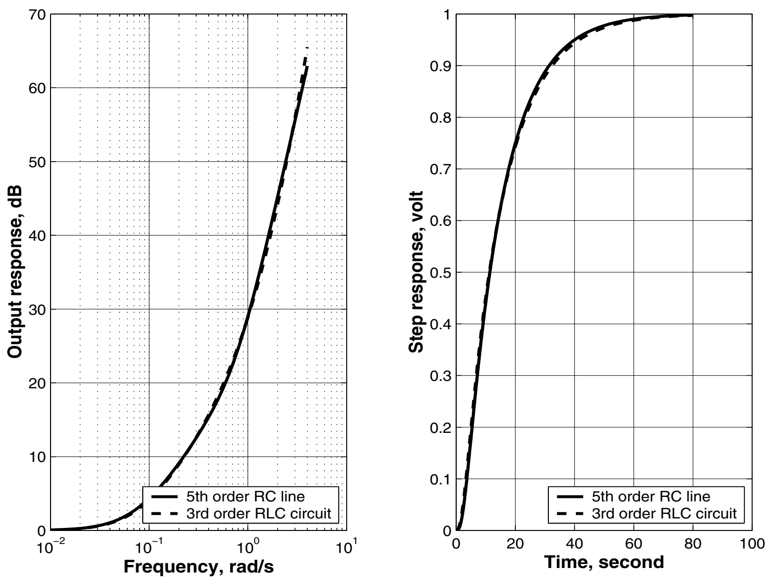


Fig. 11.7 Frequency and step responses of a fifth order *RC* line and those of a reduced order model, which is a third order *RLC* circuit. The frequency response of the third order model matches that of the fifth order circuit in the least squares sense (Problem 11.7)

Chapter 12
All-Pass Circuits

12.1 Introduction

An all-pass circuit is one whose gain is constant for all frequencies. To attain this unique property, the poles and zeros of its transfer function are placed symmetrically about the $j\omega$ axis. For each pole in the left-half s-plane, there is a zero in the right-half s-plane which is a mirror image of the pole. It follows that an all-pass circuit is non-minimum phase. By suitably locating the pole-zero pairs, we can create useful circuits in which the phase response, rather than the gain, is important.

In this chapter, we will present three examples of an all-pass circuit, first as a lumped delay line, second as a wide-band 90° phase difference circuit used in a single-sideband system, and third as a delay equalizer added to a band-pass filter so that the over-all group delay is constant (linear phase).

12.2 All-pass transfer function

Consider the circuit of Fig. 12.1. Let the all-pass transfer function $H(s)$ be written as:

$$H(s) = \frac{V_2}{E} = A\frac{P(-s)}{P(s)} \tag{12.1}$$

$$= A\prod_i \frac{(a_i - s)}{(a_i + s)} \prod_i \frac{(s^2 - b_i s + c_i)}{(s^2 + b_i s + c_i)} \tag{12.2}$$

$$= A\frac{m-n}{m+n} = A\frac{1-\dfrac{n}{m}}{1+\dfrac{n}{m}}, \tag{12.3}$$

where A is the gain, a constant; $P(s)$ a Hurwitz polynomial; a_i, b_i and c_i are positive real constants; and m and n are the even and odd parts, respectively, of $P(s)$.

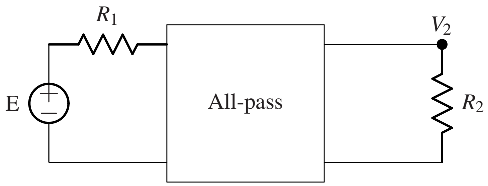


Fig. 12.1 Circuit in conjunction with study of all-pass circuits

Let the frequency response of the circuit be:

$$H(j\omega) = |H(j\omega)|e^{j\theta(\omega)} = Ae^{j\theta(\omega)}, \tag{12.4}$$

with the phase given by:

$$\theta(\omega) = -2\sum_i \arctan\frac{\omega}{a_i} - 2\sum_i \arctan\frac{b_i\omega}{c_i-\omega^2} = -2\arctan X(\omega), \tag{12.5}$$

where

$$X(\omega) = -j\frac{n}{m}\bigg|_{s=j\omega}. \tag{12.6}$$

We recall from Theorem 9.4 that n/m is an LC impedance function, so $X(\omega)$ is the reactance function of n/m.

The group delay is easily found to be:

$$D(\omega) = -\frac{d\theta}{d\omega} = 2\sum_i \frac{a_i}{a_i^2+\omega^2} + 2\sum_i \frac{b_i(c_i+\omega^2)}{(c_i-\omega^2)^2+b_i^2\omega^2}. \tag{12.7}$$

As shown in Fig. 12.2, each first order term provides a delay characteristic similar to that of a low-pass filter whose "bandwidth" is controlled by a_i, and each second order term resembles the response of a resonant circuit whose bandwidth and peak are determined by the parameters b_i and c_i.

12.3 Realizations of all-pass transfer functions

The synthesis of all-pass transfer functions is straightforward. Once the transfer function is known, it can be realized in any number of standard circuit configurations. We will present examples of LC and RC realizations.

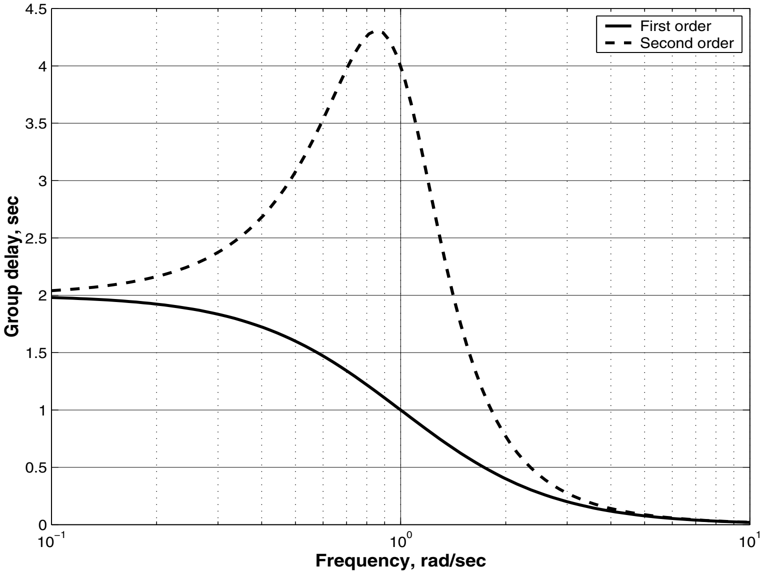


Fig. 12.2 Group delay of first and second order all-pass sections

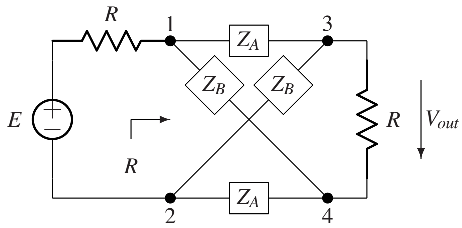


Fig. 12.3 A constant-R lattice; $Z_A(s)Z_B(s) = R^2$

12.3.1 Constant-resistance lattice

Figure 12.3 shows a constant-resistance lattice. If the condition $Z_A Z_B = R^2$ holds, the transfer function is:

$$H(s) = \frac{V_{out}}{E} = \frac{1}{2}\frac{R - Z_A}{R + Z_A}. \tag{12.8}$$

If the impedance $Z_A(s)$ is LC, the transfer function is all-pass. The most interesting property of the circuit is that the input impedance across nodes 1 and 2 is R when nodes 3 and 4 are terminated in R, so that we may cascade any number of such lattices as shown in Fig. 12.4 and the overall transfer function is simply the product of individual all-pass functions, namely,

$$H(s) = \frac{V_{out}}{E} = \frac{1}{2}\prod_i \frac{R - Z_{A_i}}{R + Z_{A_i}}\,. \tag{12.9}$$

provided in each section the condition $Z_{A_i} Z_{B_i} = R^2$ holds.

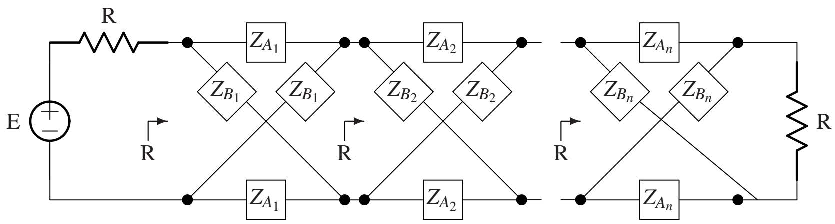


Fig. 12.4 Cascade connection of constant-resistance all-pass sections

A first order section can be realized by setting Z_A to be an inductor and Z_B a capacitor with the following relations:

$$Z_A(s) = sL\,, \qquad Z_B(s) = \frac{1}{sC}\,, \qquad C = \frac{R^2}{L}\,, \tag{12.10}$$

and the transfer function is

$$H(s) = \frac{1}{2}\frac{R - sL}{R + sL}\,. \tag{12.11}$$

A second order section can be realized by setting Z_A to be an inductor L_A in parallel with a capacitor C_A, and Z_B an inductor L_B in series with a capacitor C_B, to obtain a transfer function:

$$H(s) = \frac{1}{2}\frac{s^2 - \dfrac{1}{RC_A}s + \dfrac{1}{L_A C_A}}{s^2 + \dfrac{1}{RC_A}s + \dfrac{1}{L_A C_A}}\,, \tag{12.12}$$

and the elements are related as

$$L_B = C_A R^2 \qquad C_B = \frac{L_A}{R^2}\,. \tag{12.13}$$

The constant-resistance lattice as a two-port has one disadvantage and it is that the input and output do not have a common ground. For a second order section, the two-port is equivalent to a constant-resistance bridged-T shown in Fig. 12.5, where the elements are related to those of the lattice as follows.

$$L_1 = L_A \qquad L_2 = (C_A R^2 - L_A)/2 \qquad C_1 = C_A/2 \qquad C_2 = 2L_A/R^2 \tag{12.14}$$

It is possible that $L_2 < 0$. In that case, the three inductors can be replaced by a system of coupled inductors. It should be emphasized that the bridged-T circuit is a constant-resistance two-port. We may cascade any number of them in the same way as constant-resistance lattices.

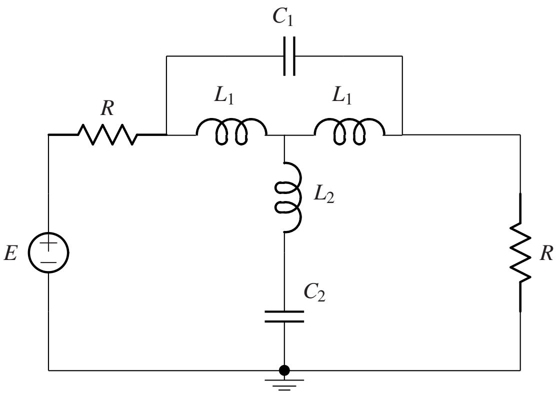
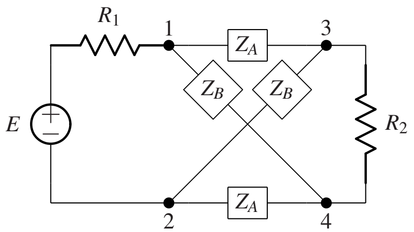


Fig. 12.5 Second order, constant-resistance bridged-T all-pass section

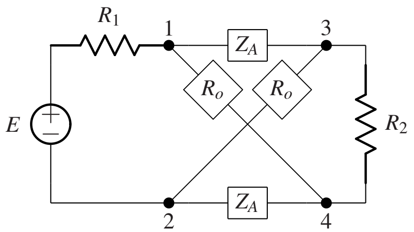
12.3.2 Non-constant-resistance lattice

If we forgo the requirement that the input impedance of a lattice be R when it is terminated in R, all-pass lattices exist that are more economical in elements [60]. The transfer function is realized in the form of Eq. (12.3). Shown in Figs. 12.6-12.8 are three such lattices and their respective transfer functions. In each case, if Z_A is an LC impedance, the circuit is all-pass.



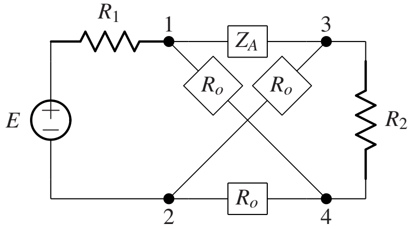
$$Z_A Z_B = R_o^2$$
$$R_1 = R_o, \quad R_2 = \eta R_o$$
$$H(s) = \frac{\eta}{1+\eta} \frac{R_o - Z_A}{R_o + Z_A}$$

Fig. 12.6 A non-constant-resistance lattice [60]



$$Z_B = R_o$$
$$R_1 = R_o/\eta, \quad R_2 = \eta R_o$$
$$H(s) = \frac{\eta}{2+\eta+1/\eta} \frac{R_o - Z_A}{R_o + Z_A}$$

Fig. 12.7 Another non-constant-resistance lattice [60]



$$R_1 = R_o/\eta, \quad R_2 = \eta R_o$$
$$H(s) = \frac{\eta}{2(2+\eta+1/\eta)} \frac{R_o - Z_A}{R_o + Z_A}$$

Fig. 12.8 A third non-constant-resistance lattice [60]

12.3.3 RC *all-pass circuits*

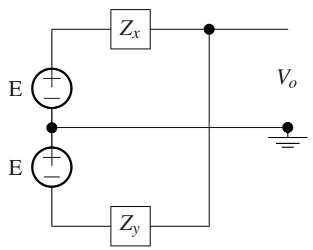
If an all-pass transfer function consists of only first order sections, and if both E and $-E$ are available (differential input), then the transfer function can be realized using resistors and capacitors only [40, 67]. With reference to Fig. 12.9, let $H(s)$ be the transfer function written as:

$$H(s) = \frac{V_o}{E} = \frac{Z_y - Z_x}{Z_y + Z_x} = K \prod_i \frac{a_i - s}{a_1 + s}, \tag{12.15}$$

where the a_i are all positive real. Re-writing this equation, we obtain

$$\frac{Z_x}{Z_y} = \frac{1 - H(s)}{1 + H(s)} = \frac{\prod_i (a_i + s) - K \prod_i (a_i - s)}{\prod_i (a_i + s) + K \prod_i (a_i - s)}. \tag{12.16}$$

Both Z_x and Z_y are to be RC impedances, so the poles and zeros of the function Z_x/Z_y must all be negative real. Weaver [67] has shown that if K is sufficiently small, it is possible to realize Z_x and Z_y as RC impedances.



$$H(s) = \frac{V_o}{E} = \frac{Z_y - Z_x}{Z_y + Z_x}$$

Fig. 12.9 An RC all-pass circuit [67]

As an example, suppose an all-pass function is given:

$$H(s) = K\frac{(1-s)(2-s)}{(1+s)(2+s)}. \tag{12.17}$$

The ratio of Z_x to Z_y is

$$\frac{Z_x}{Z_y} = \frac{(1-K)s^2+3(1+K)s+2(1-K)}{(1+K)s^2+3(1-K)s+2(1+K)}. \tag{12.18}$$

We must choose a value of K such that this function has negative real poles and zeros. One such value is

$$K = \frac{\sqrt{10}-3}{\sqrt{10}+3} = 0.026334, \tag{12.19}$$

and the ratio becomes

$$\frac{Z_x}{Z_y} = \frac{3}{\sqrt{10}}\frac{s^2+\sqrt{10}s+2}{s^2+\dfrac{9}{\sqrt{10}}s+2} = \frac{3}{\sqrt{10}}\frac{(s+2.2882)(s+0.8740)}{(s+1.5811)(s+1.2649)}. \tag{12.20}$$

Now there are many different ways to assign the poles and zeros to Z_x and Z_y such that they are RC. Each assignment will lead to a different circuit. We choose

$$Z_x = \frac{3(s+2.2882)}{(s+1.5811)} = 3+\frac{2.12132}{s+1.5811}; \tag{12.21}$$

$$Z_y = \frac{(s+1.2649)}{\sqrt{10}(s+0.8740)} = 0.31623+\frac{0.1236}{s+0.8740}. \tag{12.22}$$

It is seen that both are RC impedances.

12.4 Lumped delay line

An ideal delay line is a circuit whose gain and group delay are constant, independent of frequency. Its transfer function is in fact $Ae^{-s\tau}$, where A is the gain and τ the delay. Since the exponential function is not a rational function, it cannot be the transfer function of any RLC circuit, and we can only construct a circuit whose rational transfer function approximates the exponential function. There are many such approximations. (See [18] for a partial list. See [72] for a low-pass approximation.) We will consider one which is based on an all-pass Padé approximation of the exponential function. It has the advantage of preserving the constancy of the gain so that all the frequency components of the signal pass through the circuit un-attenuated; only their phases are changed.

Let p be the normalized frequency:

$$p = s\tau. \tag{12.23}$$

Let $H_n(p)$ be an nth order all-pass function:

$$H_n(p) = \frac{P_n(-p)}{P_n(p)} = \frac{(-1)^n p^n + (-1)^{n-1} a_{n-1} p^{n-1} + \cdots - a_1 p + a_0}{p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0} . \qquad (12.24)$$

Let the exponential function be expanded in Taylor series:

$$e^{-p} = 1 - p + \frac{1}{2!} p^2 - \frac{1}{3!} p^3 + \cdots . \qquad (12.25)$$

Equating the rational function $H_n(p)$ to the series, multiplying both sides by the denominator polynomial, collecting terms, and finally equating the coefficients of terms of like powers, we obtain for $n = 2, 3, 4$ the following Padé approximations:

$$H_2(p) = \frac{p^2 - 6p + 12}{p^2 + 6p + 12} ; \qquad (12.26)$$

$$H_3(p) = \frac{-p^3 + 12p^2 - 60p + 120}{p^3 + 12p^2 + 60p + 120} ; \qquad (12.27)$$

$$H_4(p) = \frac{p^4 - 20p^3 + 180p^2 - 840p + 1680}{p^4 + 20p^3 + 180p^2 + 840p + 1680} . \qquad (12.28)$$

The general formula for $P_n(p)$ is [18]:

$$P_n(p) = 1 + \frac{1}{2} \cdot \frac{p}{1!} + \frac{1}{2} \cdot \frac{(n-1)}{(2n-1)} \cdot \frac{p^2}{2!} + \frac{1}{2} \cdot \frac{(n-1)\cdots 2 \cdot 1}{(2n-1)\cdots(n+1)} \cdot \frac{p^n}{n!} . \qquad (12.29)$$

The delay characteristics of the three functions are shown in Fig. 12.10. To compare results, we may define a constant-delay "bandwidth" as the frequency range $[0, \omega_o]$ where ω_o is the frequency at which the delay drops to 90% of its value at DC. For $n = 2$, $\omega_o \tau$ is about 2.2, meaning the circuit provides a one-second delay over a bandwidth of about 2.2 rad/sec., or a two-second delay over 1.1 rad/sec. To get the same delay for a larger bandwidth, we may use a higher order circuit, but the bandwidth does not increase proportionately with order. In practice, it is best to realize the transfer functions by a cascade connection of second order constant-resistance lattices. Each section provides a unit delay for a specified bandwidth and we then cascade as many sections as needed to provide the desired delay.

The second order section can be realized as a bridged-T circuit in which the three inductors are replaced with two coupled inductors of self-inductance, on a one-ohm basis, $1/3\,\mathrm{H}$, and mutual inductance of $-1/6\,\mathrm{H}$, $C_1 = 1/12\,\mathrm{F}$, and $C_2 = 1\,\mathrm{F}$.

12.5 Wide-band 90° phase difference circuit

As a second example of application of all-pass circuits, we consider the design of a 90° phase difference circuit for use in communication systems.

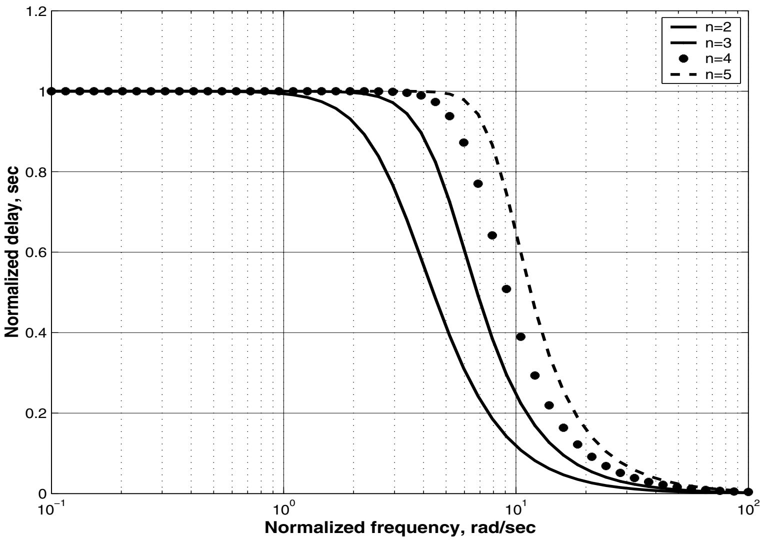


Fig. 12.10 Delay characteristics of all-pass *Padé* approximation of e^{-p}

12.5.1 Introduction

In a single-side-band communication system, the base-band signal (e.g., voice) is split into two signals of equal amplitude such that the difference of their phase angles is constant at 90° over a frequency range (e.g., 200 to 3000Hz). The two signals are modulated by two carriers in quadrature (e.g., $\sin\omega_c t$ and $\cos\omega_c t$). The modulated signals are then added to produce a lower-side-band or upper-side-band signal for transmission. The block diagram is shown in Fig. 12.11.

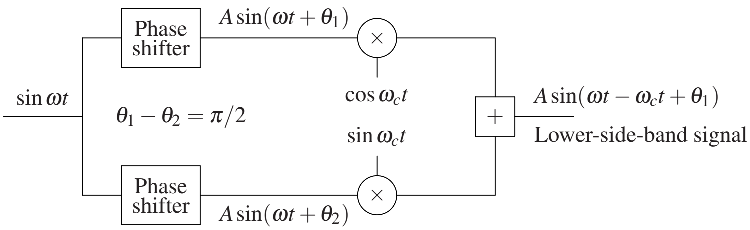


Fig. 12.11 Single-side-band transmitter system. A circuit is needed to split the input signal into two whose phase difference is 90° over a wide band

Wide-band phase difference circuits are also used in image rejection receivers [55] and in quadrature *FM* detectors [54]. Sometimes, these circuits are called phase-splitting circuits.

The design of phase splitting circuits drew the attention of many researchers in the 1940s. Darlington [20], Orchard [47], and Saraga [60] appeared to have come up with similar solutions at about the same time independently, in which the phase difference of the two signals approximates a constant in the Chebyshev sense, namely, the maximum deviation from a constant is minimized. All the solutions use elliptic functions to obtain the desired results.

The phase-splitting circuit itself consists of two all-pass circuits driven by a common signal source as shown in Fig. 12.12. The use of all-pass circuits guarantees

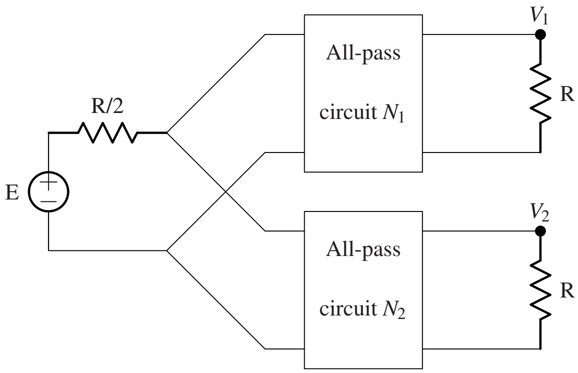


Fig. 12.12 Phase difference circuit consisting of two all-pass circuits

that the two amplitudes will be equal for all frequencies. The all-pass circuits can be realized in one or more constant-resistance lattices or non-constant-resistance lattices.

12.5.2 Formulation of problem

First we normalize the frequency variable. Let ω_1 and ω_2 be the lower and upper limits of the frequency range over which we wish the phase difference to approximate 90°. Let ω_0 be their geometric mean:

$$\omega_0 = \sqrt{\omega_1 \omega_2}\,. \tag{12.30}$$

Let the normalized complex frequency variable be:

$$p = \frac{s}{\omega_0} = \frac{\sigma}{\omega_0} + j\frac{\omega}{\omega_0} = r + jx\,. \tag{12.31}$$

The normalized lower and upper limits are:

$$x_1 = \frac{\omega_1}{\omega_0} = \sqrt{\frac{\omega_1}{\omega_2}} \equiv \sqrt{k}. \tag{12.32}$$

$$x_2 = \frac{\omega_2}{\omega_0} = \sqrt{\frac{\omega_2}{\omega_1}} \equiv \frac{1}{\sqrt{k}}. \tag{12.33}$$

The frequency range of interest extends from $x = \sqrt{k}$ to $x = 1/\sqrt{k}$. Note that the normalized frequency is symmetric with respect to $x = 1$ on a logarithmic scale.

With reference to Fig. 12.12, let the transfer function of the all-pass circuit N_1 be

$$H_1(p) = \frac{1}{2}\frac{1 - \dfrac{Z_1(p)}{R}}{1 + \dfrac{Z_1(p)}{R}}. \tag{12.34}$$

If $Z_1(p)$ is LC, then for $p = jx$ ($s = j\omega$), we have

$$H_1(jx) = \frac{1}{2}\frac{1 - j\dfrac{X_1(x)}{R}}{1 + j\dfrac{X_1(x)}{R}} = \frac{1}{2}e^{-j\theta_1(x)}, \tag{12.35}$$

where $X_1(x)$ is the reactance of $Z_1(jx)$. Similarly for the second all-pass circuit:

$$H_2(jx) = \frac{1}{2}\frac{1 - j\dfrac{X_2(x)}{R}}{1 + j\dfrac{X_2(x)}{R}} = \frac{1}{2}e^{-j\theta_2(x)}, \tag{12.36}$$

The phase angles are:

$$\frac{\theta_1}{2} = \arctan\frac{X_1(x)}{R} \qquad \frac{\theta_2}{2} = \arctan\frac{X_2(x)}{R}. \tag{12.37}$$

Let the phase difference be θ. Then we have

$$y(x) = \tan\frac{\theta}{2} = \tan(\frac{\theta_1}{2} - \frac{\theta_2}{2}) = \frac{\dfrac{X_1(x)}{R} - \dfrac{X_2(x)}{R}}{1 + \dfrac{X_1(x)}{R}\dfrac{X_2(x)}{R}}. \tag{12.38}$$

Since the phase function is an odd function of x, $y(x)$ is odd. Our problem is to find an odd function $y(x)$ such that $\theta \approx \pi/2$ or $y(x) \approx 1$ over the frequency range $[\sqrt{k}, 1/\sqrt{k}]$ in a Chebyshev manner. This is an approximation problem.

Once $y(x)$ has been found, we need to identify the two reactance functions $X_1(x)$ and $X_2(x)$, and then realize the two all-pass circuits characterized by the respective reactance functions. This is a synthesis problem.

As we will see, the phase difference function $y(p)$ derived from $y(x)$ $(x = -jp)$ is a rational function with negative real zeros and poles. As such, as shown by Weaver [67], it can be realized in a circuit of Fig. 12.12 in which the all-pass circuits are *RC*. This design is particularly suited for implementation in an integrated circuit [33].

12.5.3 Approximation problem

With reference to Eq. (12.38), we look for a solution $y(x)$ such that as x varies from $\sqrt{k}$ to $1/\sqrt{k}$ in the "constant phase difference band," y would oscillate between two constants, crossing the value one n times. The solution is found to be:

$$\frac{y}{\sqrt{\lambda}} = sn(u/M, \lambda)\,, \tag{12.39}$$

$$\frac{x}{\sqrt{k}} = sn(u,k)\,. \tag{12.40}$$

The phase difference y as a function of x is obtained indirectly through the auxiliary variable u. As x varies from $\sqrt{k}$ to$1/\sqrt{k}$, y oscillates between $\sqrt{\lambda}$ and $1/\sqrt{\lambda}$, crossing the value one n times. So $\sqrt{\lambda}$ is the maximum deviation from one (on a logarithmic scale).

The Jacobian elliptic function *sn*, and its sister functions *cn* and *dn*, are briefly described in Sect. 10.4.5. Let us remind ourselves that in $sn(u,k)$, u is its argument and its modulus is the constant k. Similarly, in $sn(u/M,\lambda)$, the argument is u/M and the modulus is λ.

We also need the complementary modulus k':

$$k' = \sqrt{1-k^2}\,, \tag{12.41}$$

and the complete elliptic integral of the first kind $K(k)$ and its complementary $K'(k) = K(k')$. Similarly for λ, λ', $K(\lambda)$ and $K'(\lambda)$.

We recall from Chapter 10 that for real values of the argument, the elliptic functions are periodic with period $4K(k)$. For imaginary values, they are periodic with period $2K'(k)$. Fig. 10.13 and Fig. 10.14 show a plot of $sn(u,k)$ and $sn(ju,k)$, respectively.

So, the function $sn(u,k)$ is a doubly periodic function. It is best to represent its behavior in a "periodic rectangle" on a complex plane $u = v + jw$, shown in Fig. 12.13. The rectangle is repeated every $4K(k)$ horizontally and every $2K'(k)$ vertically. The same rectangle can be used to represent $x = \sqrt{k}\,sn(u,k)$ and $y = \sqrt{\lambda}\,sn(u/M,\lambda)$, provided in the latter the labels are changed to $v = MK(\lambda), 2MK(\lambda), \ldots$ and $w = MK'(\lambda), 2MK'(\lambda), \ldots$ and so on.

Now concentrate on the lower left-hand sub-rectangle. As $u = v + j0$ moves from 0 to $K(k)$, x moves from 0 to $\sqrt{k}$. Then turn 90° and follow the vertical line $u = K(k) + jw$. As u moves from $K(k) + j0$ to $K(k) + jK'(k)$, x moves from $\sqrt{k}$ to

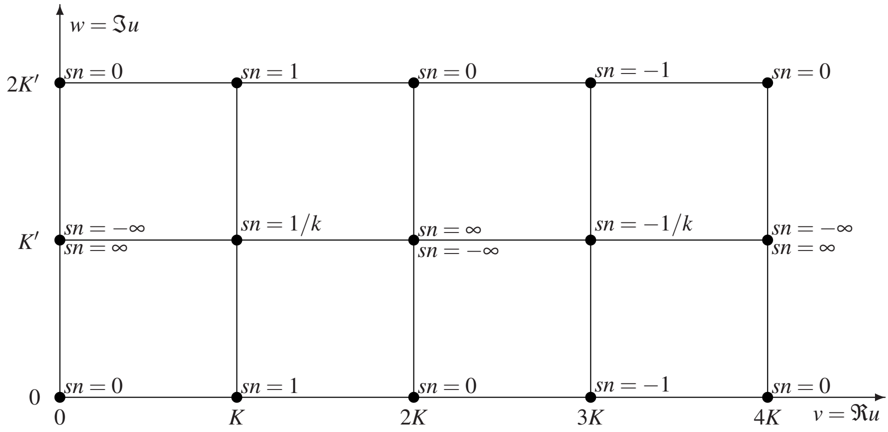


Fig. 12.13 Periodic rectangle of elliptic sine function $sn(u,k)$

$1/\sqrt{k}$, covering the constant phase difference band. Now turn 90° again and follow the horizontal line $u = v + jK'(k)$. x moves from $1/\sqrt{k}$ to $\pm\infty$.

As $u = v + jw$ varies, so does y. In the first segment, u covers a length of $K(k)$ and we want the argument of y to cover $MK(\lambda)$. So we have the relation:

$$K(k) = MK(\lambda)\,. \tag{12.42}$$

In the second segment, as x varies from $\sqrt{k}$ to $1/\sqrt{k}$, we want y to cross one n times. So we have:

$$K'(k) = nMK'(\lambda)\,. \tag{12.43}$$

The modulus λ is thus related to the modulus k as follows.

$$\frac{K'(\lambda)}{K(\lambda)} = F(\lambda) = \frac{1}{n}\frac{K'(k)}{K(k)} = \frac{1}{n}F(k)\,. \tag{12.44}$$

The three design parameters: k, which specifies the frequency range over which we want a constant phase difference, λ, which specifies the deviation of the phase difference from 90°, and n, which is the order of the approximation, are connected by the relation 12.44. This suggests that there are only two independent design parameters. In practice, we specify k and n (an integer) and find λ to satisfy 12.44. For this purpose, it is best to construct $F(k)$ for a range of k. Choose k. Find $F(k)$. Then find λ such that $F(\lambda) = F(k)/n$.

Once we have k, n and λ, we can find the poles and zeros of y. Follow the vertical line $u = 0 + jw$. We see that zeroes of y occur at

$$u_z = j2qMK'(\lambda)\,, \quad q = 0,1,2,\ldots,n-1 \tag{12.45}$$

The corresponding x values are found from

$$x_z = \sqrt{k}\, sn(j2qK'(k)/n, k) = j\sqrt{k}\, sc(2qK'(k)/n, k'), \quad q = 0, 1, \ldots, n-1. \tag{12.46}$$

Poles of y occur at

$$u_p = j(2q+1)K'(\lambda), \quad q = 0, 1, \ldots, n-1. \tag{12.47}$$

The corresponding x values are found from

$$x_p = \sqrt{k}\, sn(j(2q+1)K'(k)/n, k) = j\sqrt{k}\, sc((2q+1)K'(k)/n, k'), \quad q = 0, 1, \ldots, n-1, \tag{12.48}$$

where we have used the identity:

$$sn(ju, k) = j\frac{sn(u, k')}{cn(u, k')} = j\, sc(u, k'). \tag{12.49}$$

The poles and zeros have geometric symmetry. In fact, $y(x)$ for various n, denoted as $y_n(x)$, can be written as [60]:

$$y_1 = x; \qquad y_2 = \frac{Hx}{1+x^2}; \qquad y_3 = x\,\frac{a+x^2}{1+ax^2}; \tag{12.50}$$

$$y_4 = \frac{Hx(1+x^2)}{(1+ax^2)(1+\frac{1}{a}x^2)}; \tag{12.51}$$

$$y_5 = x\,\frac{(a+x^2)(b+x^2)}{(1+ax^2)(1+bx^2)}; \tag{12.52}$$

$$y_6 = \frac{Hx(1+ax^2)(1+\frac{1}{a}x^2)}{(1+x^2)(1+bx^2)(1+\frac{1}{b}x^2)}. \tag{12.53}$$

It should be noted that if we replace x by $1/x$, y becomes $1/y$ if n is odd; y is unchanged if n is even. The multiplicative constant H for even n is found as follows.

$$H = \lim_{x\to 0}\frac{y}{x} = \frac{\sqrt{\lambda}}{\sqrt{k}}\lim_{u\to 0}\frac{sn(u/M, \lambda)}{sn(u, k)} = \frac{\sqrt{\lambda}}{M\sqrt{k}} = n\frac{\sqrt{\lambda}K'(\lambda)}{\sqrt{k}K'(k)}, \tag{12.54}$$

where we have used the relation:

$$\frac{d}{du}sn(u, k) = cn(u, k)\, dn(u, k). \tag{12.55}$$

12.5.4 Synthesis problem

Once we have $y(x)$, we follow a procedure developed by Darlington to identify the reactance functions $X_1(x)$ and $X_2(x)$ [20].

We recall that $y(x) = \tan(\theta/2)$ is an odd function of x. It can be written as

$$\tan\frac{\theta}{2} = \frac{xB}{A}, \tag{12.56}$$

where A and B are even polynomials of x. This suggests that

$$\frac{\theta}{2} = arg(A + jxB), \tag{12.57}$$

where $arg(z)$ is the argument of the complex quantity z.

Similarly, θ_1 and θ_2 of Eq. (12.37) can be written as

$$\frac{\theta_1}{2} = arg(A_1 + jxB_1), \qquad \frac{\theta_2}{2} = arg(A_2 + jxB_2), \tag{12.58}$$

where A_1, A_2, B_1 and B_2 are even polynomials in x. We now note that

$$-\frac{\theta_2}{2} = arg(A_2 - jxB_2), \tag{12.59}$$

and we have the important relation:

$$\frac{\theta}{2} = \frac{\theta_1}{2} - \frac{\theta_2}{2} = arg[(A_1 + jxB_1)(A_2 - jxB_2)]. \tag{12.60}$$

It follows that

$$(A + jxB) = C(A_1 + jxB_1)(A_2 - jxB_2), \tag{12.61}$$

where C is a real constant.

Once we have $y(x)$, we take it back to the complex frequency plane where it came from. That is, find $y(p)$ such that

$$y(p) = jy(x)|_{x=-jp} = \frac{pB(p^2)}{A(p^2)}, \tag{12.62}$$

and form the polynomial

$$Q(p) = A + pB. \tag{12.63}$$

Factor $Q(p)$ into $Q_1(p)Q_2(p)$ where $Q_1(p)$ contains all the roots with negative real parts and $Q_2(p)$ contains all the roots with positive real parts. The polynomials A_1, B_1, A_2 and B_2 are found from:

$$Q_1(p) = A_1 + pB_1, \tag{12.64}$$

$$Q_2(p) = A_2 - pB_2. \tag{12.65}$$

Since $Q_1(p)$ and $Q_2(-p)$ are Hurwitz by construction, pB_1/A_1 and pB_2/A_2 are LC impedances and we can make the identifications:

$$\frac{Z_1(p)}{R} = \frac{pB_1(p^2)}{A_1(p^2)}, \tag{12.66}$$

$$\frac{Z_2(p)}{R} = \frac{pB_2(p^2)}{A_2(p^2)}, \tag{12.67}$$

for the two all-pass transfer functions:

$$H_1(p) = \frac{1}{2}\frac{1 - \dfrac{Z_1(p)}{R}}{1 + \dfrac{Z_1(p)}{R}}, \tag{12.68}$$

$$H_2(p) = \frac{1}{2}\frac{1 - \dfrac{Z_2(p)}{R}}{1 + \dfrac{Z_2(p)}{R}}, \tag{12.69}$$

and proceed to realize the all-pass circuits of Fig. 12.12, by any means given in Sect. 12.3. This completes the synthesis of the 90° phase difference circuit.

12.5.5 Example

Let us design a 90° phase difference circuit to cover the voice band, which ranges from 200 Hz to 3000 Hz. Let the order of the approximation be 4 and 6 and compare results.

We will give the design details for $n = 4$. Those for $n = 6$ are entirely similar.

1. Compute k, k', $K(k)$ and $K'(k)$.

$$k = \frac{\omega_1}{\omega_2} = \frac{200}{3000} = 0.06666667\,.$$
$$k' = \sqrt{1-k^2} = 0.99777530\,.$$
$$K(k) = 1.57254603\,.$$
$$K'(k) = 4.09779088\,.$$

2. For $n = 4$, find λ such that

$$\frac{K'(\lambda)}{K(\lambda)} = \frac{K'(k)}{nK(k)}$$
$$\lambda = 0.93760514\,, \quad \sqrt{\lambda} = 0.96830013\,.$$

At this point, we know the phase difference θ will oscillate between $\sqrt{\lambda}\pi/2$ and $\pi/2/\sqrt{\lambda}$, i.e.,

$$87.147^\circ \le \theta \le 92.946^\circ\,. \tag{12.70}$$

If this range is acceptable, we go on to complete the design. If not, we increase the order of approximation n by one or more. Let us continue with the design.

3. Using Eqs. (12.46) and (12.48), we find the zeros and poles of the phase difference function y to be:

$$\begin{array}{llll} \text{Zeros:} & z_1 = 0 & \text{Poles:} & p_1 = +j0.31366651 \\ & z_2 = +j & & p_2 = +j3.18809940 \\ & z_3 = j\infty & & p_3 = -j3.18809940 \\ & z_4 = -j & & p_4 = -j0.31366651 \end{array}$$

4. The multiplicative constant H is, by Eq. (12.54),

$$H = 5.93694537\,, \tag{12.71}$$

and the phase difference function is

$$y(x) = \frac{H\,x(x^2+1)}{x^4 + 10.26236447x^2 + 1}\,, \tag{12.72}$$

which, in the complex domain, is:

$$y(p) = \ jy(x)|_{x=-jp} = \frac{H\,p(-p^2+1)}{p^4 - 10.26236447p^2 + 1} = \frac{pB(p^2)}{A(p^2)}\,. \tag{12.73}$$

5. Form $Q(p)$, $Q_1(p)$ and $Q_2(p)$:

$$\begin{aligned} Q(p) =& A(p^2) + pB(p^2) \\ =& (p + 1.73874727)(p + 0.13814612)\ldots \\ & (-p + 7.23871205)(-p + 0.57512671)\,. \end{aligned} \tag{12.74}$$

$$Q_1(p) = p^2 + 1.87689339p + 0.24020119\,. \tag{12.75}$$

$$Q_2(p) = p^2 + 7.81383876p + 4.16317666\,. \tag{12.76}$$

6. The impedances of the all-pass circuits are, on a one-ohm basis:

$$Z_1(p) = \frac{1.87689339p}{p^2 + 0.24026119}\,, \qquad Z_2(p) = \frac{7.81383876p}{p^2 + 4.16317666}\,. \tag{12.77}$$

Each is a parallel LC impedance.

7. This completes the design. The phase difference is shown in Fig. 12.14, in which we have also shown the result for $n = 6$. In the latter, the phase difference varies from 89.74° and 90.26°, which is quite acceptable.

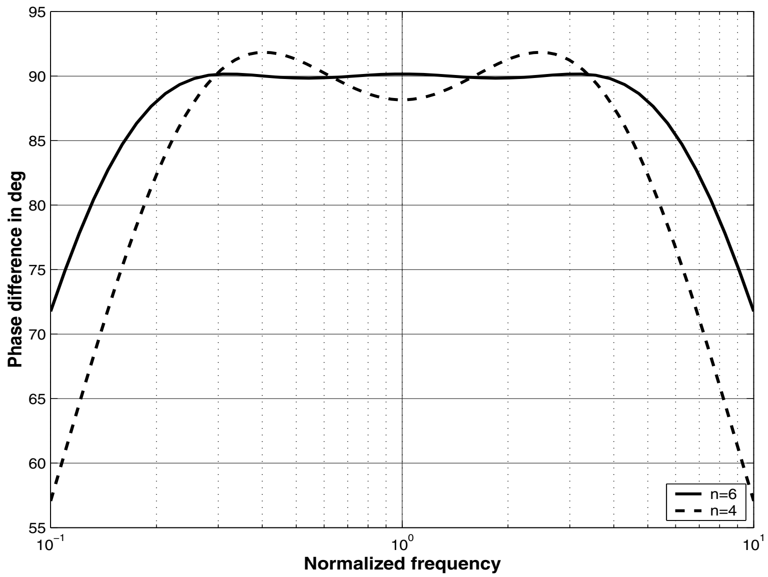


Fig. 12.14 Phase characteristics of 90° phase difference circuits of order 4 and 6

12.6 Delay Equalizer

In Chapter 4, we explained why it is important to have linear phase, or constant group delay, in a communication system. After a filter is designed to satisfy the gain or loss characteristics, we generally find that the delay is not constant, and this will cause distortion of the signal. In practice, delay equalizers in the form of all-pass circuits are added to the filter. The all-pass circuits do not change the overall gain characteristic but they add delay to that of the filter such that the overall delay is constant, at least in the passband. Outside the passband, the signal is highly attenuated anyway and delay distortion does not matter.

12.6.1 Delay equalization of a band-pass filter

Figure 12.15 shows a band-pass filter. Its gain and delay characteristics are shown in Fig. 12.16. It is seen that the gain has the desired response but its delay is not constant in the passband. To obtain a satisfactory delay performance, we connect two second-order constant-resistance all-pass circuits to the filter to correct the delay, as shown in Fig. 12.17. If we set the reference resistance R of the all-pass circuits equal to the terminating resistance of the filter, the overall delay will be approximately the sum of the delay of the bandpass filter and the delays of the two all-pass circuits.

The sum does not equal exactly the overall delay. The reason is that the all-pass circuits must be driven by a voltage source whose internal resistance equals R, but the output impedance of the band-pass filter is not R for all frequencies.

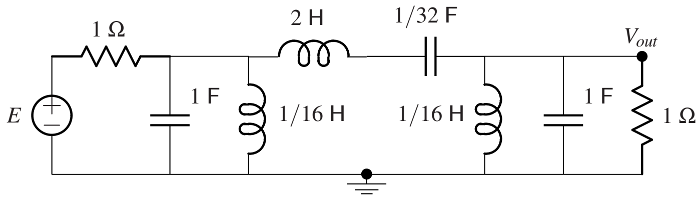


Fig. 12.15 A third order band-pass filter

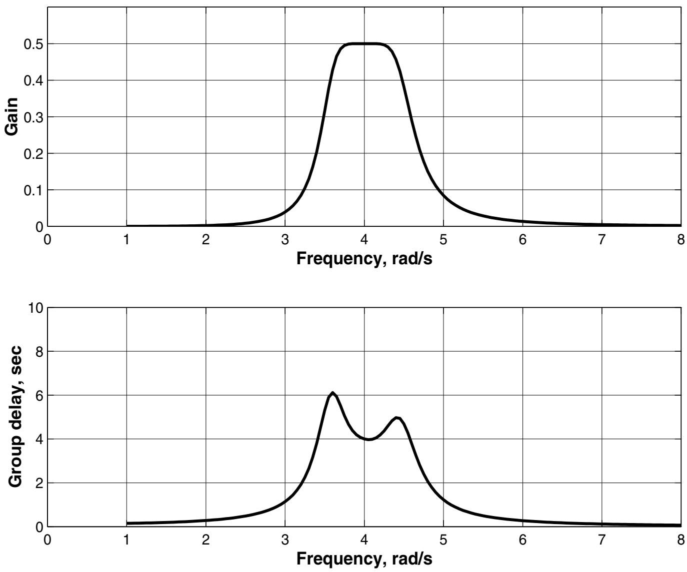


Fig. 12.16 Gain and delay characteristics of a band-pass filter

With reference to Fig. 12.2, we recall that the "peak" and "width" of the delay characteristic of a second-order all-pass circuit can be adjusted by varying its parameters "b" and "c" (see Eq. (12.7)). By a proper choice of the parameters, the delays of the all-pass circuits, when added to the delay of the band-pass filter, could give us an overall delay which is constant over the pass-band of the filter. For this purpose, we formulate the design problem as an optimization problem and let the computer program take us to the optimum solution.

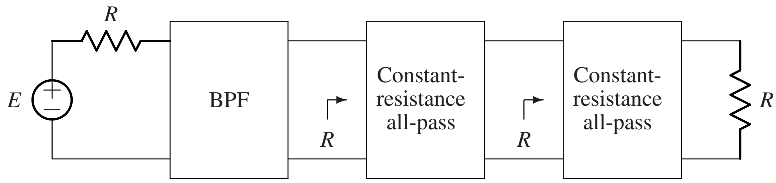


Fig. 12.17 Adding all-pass circuits to a band-pass filter to equalize delay

12.6.2 Design by optimization

Let $\mathbf{x}$ be the vector of design parameters. In our case,

$$\mathbf{x} = [b_1, c_1, b_2, c_2]. \tag{12.78}$$

Let the overall delay over a frequency range $\Omega = (\omega_\ell, \omega_u)$ be $D(\mathbf{x}, \omega)$, which can be computed from the overall circuit by node analysis. Let the average delay over Ω be

$$D_a(\mathbf{x}) = \frac{1}{m} \sum_{\omega \varepsilon \Omega} D(\mathbf{x}, \omega), \tag{12.79}$$

where m is the number of frequency points in Ω. Define a vector of functions:

$$F(\mathbf{x}) = \begin{bmatrix} D(\mathbf{x}, \omega_1) - D_a(\mathbf{x}) \\ D(\mathbf{x}, \omega_2) - D_a(\mathbf{x}) \\ \cdots \\ D(\mathbf{x}, \omega_m) - D_a(\mathbf{x}) \end{bmatrix}. \tag{12.80}$$

Finally, define the objective function $\varphi(\mathbf{x})$ as

$$\varphi(\mathbf{x}) = [F(\mathbf{x})]^T F(\mathbf{x}). \tag{12.81}$$

Our problem is to find $\mathbf{x}^*$ such that

$$\varphi(\mathbf{x}^*) = \min_{\mathbf{x}} \varphi(\mathbf{x}) \tag{12.82}$$

In effect, we are minimizing the variance of the overall delay over the pass-band of the filter.

The minimization problem is solved by the least squares method of Sect. 11.5. Starting with an initial guess: $\mathbf{x}_0 = [1.1364, 17.64, 1.25, 16.0]$, we find after 50 iterations, the objective function does not decrease, and we declare the solution at that point to be the optimized solution: $\mathbf{x}^* = [1.7428, 22.4414, 1.0071, 16.1646]$. The overall delay is shown in Fig. 12.18. It is seen that the delay in the pass-band, which is defined as $(3.55, 4.45)$, is ”constant” with a maximum deviation less than 10% of the average delay in the pass-band.

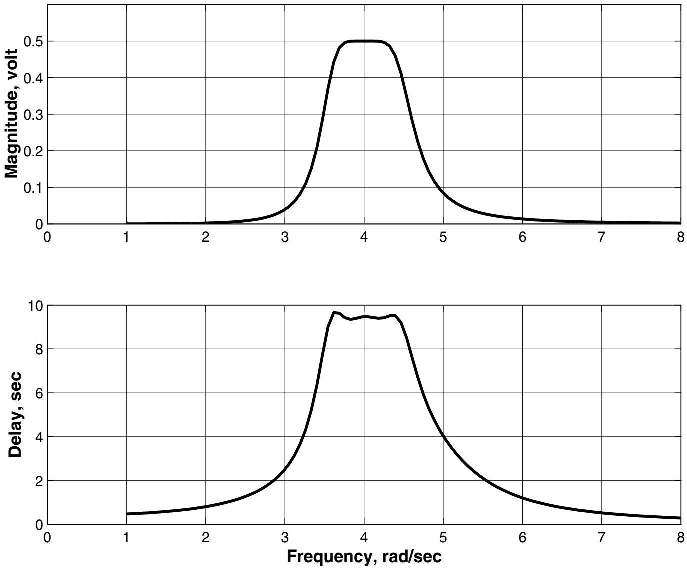


Fig. 12.18 Optimized delay of overall system consisting of a band-pass filter and two all-pass delay equalizers

The all-pass circuits are realized as constant-resistance bridged-T circuits of Fig. 12.5. Note that the overall gain remains unaffected by the addition of the all-pass circuits.

12.7 Summary

In this chapter, we have presented examples of all-pass circuits as lumped delay lines, as components of a 90° phase difference circuit, and as delay equalizers. Constant-resistance all-pass circuits have the desirable property that the transfer function of n such circuits connected in cascade is the product of the n individual transfer functions, and that the overall phase or delay is the sum of the individual phases and delays. This property allows us to make incremental extension or improvement of performance. For example, if more delay in a lumped delay line is required, we simply add more delay sections to the design. We do not have to redesign the whole circuit, as we do in the case of insertion loss in filters. If one delay equalizer is not adequate to correct a given delay characteristics, we add another. The added cost in the optimization process is minimal.

Problems

12.1. Show that the bridged-T circuit of Fig. 12.5 is equivalent to a second-order constant-resistance lattice.

12.2. Derive the transfer functions of the three non-constant-resistance lattices of Sect. 12.3.2.

12.3. In Eq. (12.17), find another value of K such that the all-pass transfer function can be realized as that of an RC circuit. What is the maximum allowable value of K? See [67].

12.4. In the Padé approximation of e^{-p}, compare the delay response of two second-order sections connected in cascade with the response of one fourth-order section. Which one has larger delay-bandwidth product?

12.5. Design a Padé lumped delay line to provide a delay of 10 nanosecond at 90% bandwidth of 100 MHz. Realize the circuit in a cascade of constant-resistance all-pass bridged-Ts of the type of Fig. 12.5.

12.6. To increase the bandwidth of "constant" delay of a lumped delay line, we can cascade a number of all-pass sections whose poles are equally spaced along the line $\sigma = -a < 0$ in the complex plane. The corresponding zeros are placed along $\sigma = a > 0$ in the right-half plane. This design is known as the "parallel plates" delay line. Let the range of frequency of interest be $-\omega_o \le \omega \le \omega_o$. Let the poles be $2\omega_o/(N-1)$ apart, where N is the total number of poles. Plot the delay as a function of frequency for various values of N for a normalized value of $\omega_o = 1$. How should we choose a? Compare the delay of this design with the delay of an all-pass circuit consisting of the same number of identical second order sections of the Padé design. If we relax the requirement that the poles be equally spaced, would you place them closer to the ends of the plates or closer to the origin?

12.7. Compute the pulse response of a second order Padé section to provide one-second delay at low frequencies by numerical integration of its node equations in the time domain. Use the backward Euler method. Realize the section in a constant-resistance all-pass bridged-T circuit of Fig. 12.5, on a one-ohm basis. Try two different input pulses, once with a narrow pulse of $E_1(t)$ and once with a broad pulse of $E_2(t)$, which are specified below.

$$E_1(t) = \begin{cases} 0.5(1+\sin((2t-0.5)\pi)), & \text{if } 0 < t \le 1; \\ 0, & \text{otherwise.} \end{cases} \tag{12.83}$$

$$E_2(t) = \begin{cases} 0.5(1+\sin((0.5t-0.5)\pi)), & \text{if } 0 < t \le 4; \\ 0, & \text{otherwise.} \end{cases} \tag{12.84}$$

The pulse responses should be as shown in Fig. 12.19. The narrow pulse $E_1(t)$ consists of high frequency components, which are delayed less than the low frequency components. The result is that the output pulse is distorted and the apparent delay is less than one second. In the second case, the pulse $E_2(t)$ consists of relatively low frequency components and they have essentially the same delay. The output is a pulse of similar shape delayed by one-second.

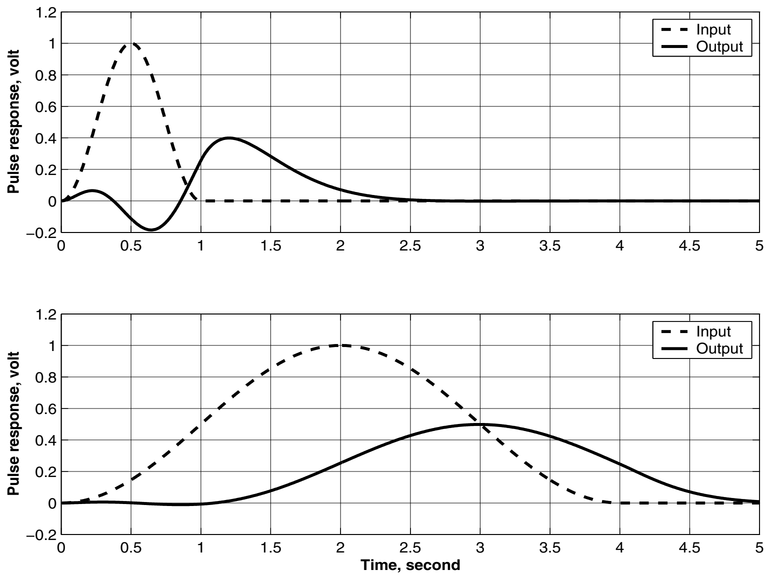


Fig. 12.19 **Upper figure**: Response of a one-second lumped delay section to a narrow pulse, showing that the output is distorted because the high frequency components are delayed less than the low frequency components. **Lower figure**: Response to a broad pulse, which consists of relatively low frequency components, is undistorted and delayed by one second, as required.

12.8. Design a 90° phase difference circuit to cover the audio band: 100 Hz to 10,000 Hz. The maximum absolute deviation from 90° is required to be less than 0.5°.

12.9. Use Weaver's *RC* all-pass circuit [67] to realize the 90° phase difference circuit of the example of Sect. 12.5.5.

12.10. Use three all-pass sections to equalize the delay distortion of the bandpass filter of Sect. 12.6.1. Extend the constant delay band to cover the range from 3.5 rad/s to 4.6 rad/s. If you have access to Optimization Toolbox of MATLAB®, use the minimax function to find an optimal solution.

Appendix A
Useful MATLAB®functions

The following are useful MATLAB®functions for circuit analysis and synthesis. The reader is encouraged to consult the reference guide and the "Help" directory in any MATLAB®window for a detailed description of each function.[1]

Ax=b: Linear equation **Ax=b** is solved by invoking $x = A\backslash b$. See also **linsolve**.
Bode plot: **[mag, phs, w]=bode(n, d, w)** places the magnitude in vector *mag* and phase in vector *phs* at each frequency in vector w of a transfer function whose numerator polynomial is n and denominator polynomial is d.
Convolution: **y=conv(p,q)** obtains the product of two polynomials, represented by their vectors of coefficients p and q.
Eigenvalues: **[V,D]=eig(A)** places the eigenvalues of A in a diagonal matrix D and their corresponding eigenvectors in V.
Elliptic functions: **[sn cn dn]=ellipj(u,m)** yields $sn(u,k)$, $cn(u,k)$, and $dn(u,k)$; and **[K, E]=ellipke(m)** yields $K(k)$ and $E(k)$, where $m = k^2$.
Factors: **y=factor(f)** obtains the factors of a symbolic polynomial f.
Imaginary part: **y=imag(z)** returns the imaginary part of z.
Linear equations: **y=linsolve(A,b)** obtains the symbolic or numerical solution of linear equation $Ax = b$, A and b being numeric or symbolic.
Matrix exponential: **H=expm(At)**, where A is a matrix of simple rational numbers and t declared symbolic, will obtain the impulse response matrix $H(t)$ in analytic form.
Numerator and denominator: **[n,d]=numden(f)** places the numerator of symbolic rational function f in n and the denominator in d.
Ordinary differential equations: **dsolve(...)** obtains the symbolic solution of an ordinary differential equation.
Polynomial from its roots: **p=poly(r)** obtain the coefficients of polynomial p in descending order in a row vector whose *roots* are placed in a column vector r.

[1] The Academia area of The MathWorks Web site provides many resources for professors and students. Check there for contributed course materials, recommended products by curriculum, tutorials on MATLAB®software, and information on MATLAB®Student Version: www.mathworks.com/products/academia/

Polynomial to symbolic: **p=poly2sym(q,'s')** converts a vector of coefficients of a polynomial q to its symbolic representation in s.

Random numbers: $\mathbf{y} = \mathbf{m} + \sigma * \mathbf{randn}$ generates a random number taken from a universe of numbers which are normally distributed with mean m and variance σ^2 .

Random numbers: **y=a + (b-a)*rand** generates a random number taken from a universe of numbers which are uniformly distributed over $[a,b]$.

Real part: **y=real(z)** returns the real part of z.

Residues: **[r,p,k]=residue(b,a)** obtains the partial fraction expansion of rational function b/a where a and b are expressed as vectors of coefficients of polynomials a and b. The residues are placed in vector r and the poles in vector p. Vector k contains the coefficients of any excess polynomial.

Roots of a polynomial: **r=roots(p)** obtains a column vector of roots of polynomial p expressed as a row vector of coefficients in descending order.

Step response: **y=step(num,den,t)** produces the step response of a transfer function whose numerator polynomial is *num* and denominator polynomial is *den*, both expressed as vectors of coefficients, for a vector of time points in t.

Simplify: **y=simplify(f)** obtains a simplified version of a symbolic function f.

Substitution: **y=subs(f, 's', w)** replaces each occurrence of s by each member of vector w in the symbolic function f. This function is useful in computing the frequency response of a rational function f if w is a vector of frequency points.

Symbolic to polynomial: **q=sym2poly(p)** converts a symbolic representation of a polynomial p to its representation as a vector q of coefficients.

Transfer function: **[num,den]=ss2tf(A,B,C,D,k)**, where A,B,C,D are the matrices of a state equation in normal form, with an input component $U_k = 1$ and all others being zero, will return a vector of transfer functions whose numerator polynomials are in the vector *num* and whose denominator polynomial, being the same for all, is in *den*.

Transfer function to zeros and poles: **[z,p,k]=tf2zp(num,den)** places the zeros of a transfer function with numerator *num* and denominator *den* in vector z, the poles in vector p and the gain in k. The vector *num* contains the coefficients of the numerator polynomial and *den* those of the denominator polynomial.

Zeros and poles to transfer function: **[num,den]=zp2tf(z,p,k)** forms the transfer function with numerator polynomial *num* and denominator polynomial *den* from a vector of zeros z, a vector of poles p and a multiplicative constant k.

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